Gamelin IX.1 # 2, #3, IX.2 #2, 3, 4

IX.1 #2 Suppose $f(z)$ is analytic, and $|f(z)| \le 1$ $\forall\ |z|<1$
Show that if $f(z)$ has a zero of order m at z_0, then $|z_0|^m \geqslant f(0)$.

Pf: We first create a function that vanishes at z_0.

$$\psi_{z_0}(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$

and $\forall\ |z|=1$, $\left|\psi_{z_0}(z)\right| = 1$.

Then, we define $F(z) = \dfrac{f(z)}{\psi_{z_0}(z)^m}$ $\forall\ |z|<1$, $z \neq z_0$.
since $f(z)$ has an order m zero at $z = z_0$, $F(z)$ has a removable singularity at $z = z_0$.

$$\sup_{|z|<1} |F(z)| = \lim_{r\to 1} \sup_{|z|=r} \frac{|f(z)|}{|\psi_{z_0}(z)|^m} \leqslant \lim_{r\to 1} \frac{\sup_{|z|=r} |f(z)|}{\inf_{|z|=r} |\psi_{z_0}(z)|^m}$$

$$\leqslant \frac{\lim_{r\to 1} \sup_{|z|=r} |f(z)|}{\lim_{r\to 1} \inf_{|z|=r} |\psi_{z_0}(z)|^m} \leq \frac{1}{1} = 1$$

Thus, $|f(z)| = |F(z)| \cdot |\psi(z)|^m \leqslant |\psi(z)|^m$ $\forall\ |z|<1$.
In particular

$$|f(0)| \leqslant |\psi(0)|^m = |z_0|^m.$$

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Alternative proof: let $\psi_{z_0}(z)$ be as before, consider

$$F(z) = f \circ \psi_{z_0}^{-1} = f \circ \psi_{-z_0}, \quad \text{then}$$

$$F(0) = f(\psi_{-z_0}(0)) = f(z_0) = 0.$$

And $F(z)$ has an order m zero at 0.

Consider $G(z) = F(z)/z^m$, and follow through the proof of Schwarz Lemma, $|F(z)| \le |z|^m$. $\forall\ |z|<1$. Hence

$$|F(z_0)| \le |z_0|^m.$$

$$F(z_0) = f(\psi_{z_0}^{-1}(z_0)) = f(0), \quad \therefore \quad |f(0)| \le |z_0|^m.$$

#3 Suppose $f(z)$ is analytic for $|z| \le 1$, and suppose $1 < |f(z)| < M$ for $|z|=1$, while $f(0)=1$. Show that $f(z)$ has a zero in the unit disk, Show that any such zero satisfies $|z_0| > \frac{1}{M}$.

Pf: (a) Consider the image of $f(\partial D)$, it is a contour contained in $\{1 < |w| < M\}$. Hence, $\forall\ |a| \le 1$, the integral

$$I(a) = \int_{f(\partial D)} \frac{1}{w-a}\, dw = \int_{\partial D} \frac{f'(z)}{f(z)-a}\, dz$$

is well defined and independent on a. By argument principle, and analyticity of f, we have

$$I(a) = \#\text{ of solution to } f(z) = a \text{ for } z \in D.$$

Since $I(1) > 0 \quad \therefore \quad I(0) = I(1) > 0.$

Thus, there must be a zero z_0 of $f(z)$, for $z_0 \in D$.

(b) Let $F(z) = f(z)/M$, then

$$F: D \to D$$

and F extends to $\overline{D}$ as a hol'c function, $F(0) = \frac{1}{M}$

We may apply problem #2 to get

$$|F(0)| \le |z_0|$$

Hence $$\frac{1}{M} \le |z_0|$$

Since for $|z|=1$, $|F(z)| < 1$, we know $F(z)$ is not an automorphism of D, thus above inequality is strict. Hence
$$\frac{1}{M} < |z_0|.$$

IX.2 #2 Show that $F(z) = \dfrac{1+3z^2}{3+z^2}$ is a finite Blashke product.

Pf: $F(z)$ vanishes at $z = \pm \frac{i}{\sqrt{3}}$, we try

$$B_{\frac{i}{\sqrt{3}}}(z) \cdot B_{-\frac{i}{\sqrt{3}}}(z) = \frac{z - \frac{i}{\sqrt{3}}}{1 + \frac{i}{\sqrt{3}}z} \cdot \frac{z + \frac{i}{\sqrt{3}}}{1 - \frac{i}{\sqrt{3}}z} = \frac{z^2 + \frac{1}{3}}{1 + \frac{1}{3}z^2}$$
$$= \frac{3z^2+1}{3+z^2}$$

OK, it worked.

#3. Suppose $f(z)$ is analytic for $|z|<3$. If $|f(z)| \le 1$,

and $f(\pm 1) = f(\pm i) = 0$, then what is the maximum for

$|f(0)|$? For which function is the max achieved ?

Pf: Let $F(z) = f(3z)$, then
$F: \mathbb{D} \to \mathbb{D}$ hol'c
with $F(\pm i/3) = F(\pm 1/3) = 0$.

Let $G(z) = B_{\frac{i}{3}}(z) \cdot B_{-\frac{i}{3}}(z) \cdot B_{\frac{1}{3}}(z) \cdot B_{-\frac{1}{3}}(z)$, where

$B_a(z) = \dfrac{z-a}{1-\bar{a}z}$ is the Blaschke factor. Then,

$\dfrac{F(z)}{G(z)} = H(z)$ is hol'c, and $\sup_{|z|<1} |H(z)| \le 1$. Hence

$|F(z)| \le |G(z)|$. And only if $H(z) = c$ constant, $|c|=1$, do we have equality.

$$|F(0)| \le |G(0)| = \left(\tfrac{1}{3}\right)^4$$

#4: Fix $z_0, z_1 \in \mathbb{D}$, find the maximum value of $|f(z_0) - f(z_1)|$ for all $f: \mathbb{D} \to \mathbb{D}$ hol'c.

Pf: Following the hint, we first prove that if $z_0 = r$, $z_1 = -r$ then only rotation, $f(z) = e^{i\theta} \cdot z$ will maximize the distance.

For any $f: \mathbb{D} \to \mathbb{D}$, let
$$g(z) = [f(z) - f(-z)]/2$$
then $g(0) = (f(0) - f(0))/2 = 0$
and $\forall |z|<1$, $|g(z)| \le \frac{1}{2}(|f(z)| + |f(-z)|) < \frac{1}{2}(1+1) = 1$.

Hence $g: \mathbb{D} \to \mathbb{D}$ and $g(0)=0$. We apply Schwaz Lemma, and get

$$|g(z)| \leq |z| \Rightarrow$$

Furthermore

$$|g(z_0) - g(z_1)| = \left| \frac{f(z_0)-f(-z_0)}{2} - \frac{f(z_1)-f(-z_1)}{2} \right|$$

$$= |f(z_0) - f(z_1)|$$

Hence $|f(z_0) - f(z_1)| = |g(z_0) - g(z_1)| \leq |g(z_0)| + |g(z_1)|$
$\leq |z_0| + |z_1|$

Plug in $z_0 = r$, $z_1 = -r$, we get
$|f(z_0) - f(z_1)| \leq 2r$

If equal sign holds, then $|g(r)| = |r|$, Hence g is in $Aut(\mathbb{D})$.
$g(z) = e^{i\theta} \cdot z$. for some θ

Hence $f(z) - f(-z) = 2e^{i\theta} \cdot z$.

We write $f(z) = g(z) + h(z)$, where $h(z) = \frac{f(z)+f(-z)}{2}$ is an even function. We claim $h(z) = 0$. (for simplicity, I will assume $f(z)$ is defined on $\overline{\mathbb{D}}$.) If $\exists z_0 \in \partial\mathbb{D}$, s.t. $h(z_0) \neq 0$, then $h(-z_0) = h(z_0)$ (h is even)

$$f(z_0) = e^{i\theta} z_0 + h(z_0)$$
$$-f(-z_0) = e^{+i\theta} z_0 - h(z_0)$$

[Lemma: if a, b are 2 complex numbers. with $|a|=1$. $|a+b|=1$, $|a-b|=1$. Then $b=0$.

Pf : $|a+b|^2 = a^2+b^2+2a\cdot b$

$|a-b|^2 = a^2+b^2-2a\cdot b$

$\therefore \quad |a+b|^2 + |a-b|^2 = 2(a^2+b^2)$

$\therefore \quad b^2 = 0$.

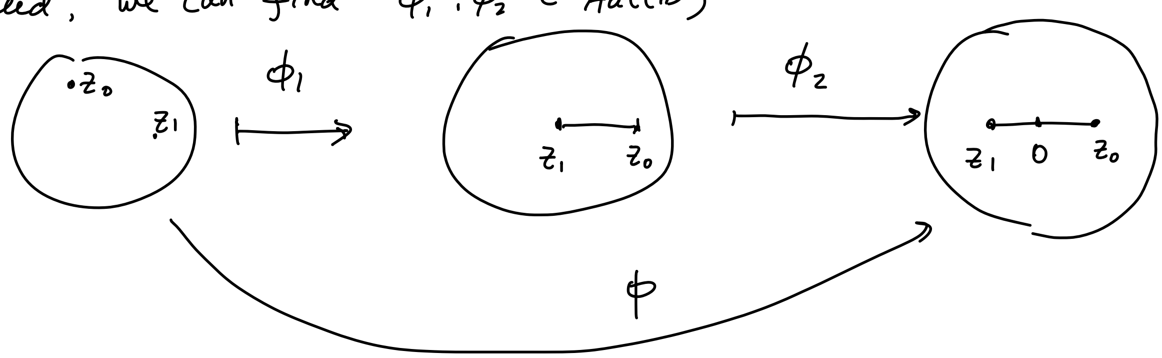
apply the Lemma to $|z_0|=1$, $|z_0+e^{-i\theta}h(z_0)|=1$

$|z_0-e^{-i\theta}h(z_0)|=1$

we get $h(z_0)=0$

Thus $f(z)$ has to be a rotation as well, if $g(z)$ is a rotation. This finishes the proof of the Hint.

Now, given any z_0, z_1, we claim there exist a $r>0$. and an $Aut(\mathbb{D})$, that sends $z_0 \mapsto r$, $z_1 \mapsto -r$. Indeed, we can find $\phi_1, \phi_2 \in Aut(\mathbb{D})$



to realize the above transformation. Finding f to max $|f(z_0)-f(z_1)|$, is equivalent to find $\tilde{f}$ to max $|\tilde{f}\circ(\phi(z_0)) - \tilde{f}(\phi(z_1))|$, since $f=\tilde{f}\circ\phi$, $\tilde{f}=f\circ\phi^{-1}$