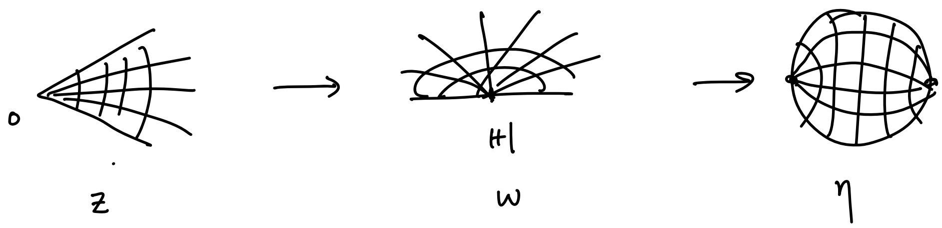
1. Find a conformal map of the sector $\{|\arg z| < \pi/3\}$ onto the open unit disk mapping 0 to -1 and ∞ to $+1$. Sketch the images of radial lines and of arcs of circles centered at 0. Is the map unique?

HW #11

Gamelin

- we first map the sector to upper half plane



$\boxed{w = i z^{\frac{3}{2}}}$, indeed. if $z = e^{i\theta}$, then

$$w = e^{\frac{\pi}{2} i} \cdot e^{i \frac{3}{2}\theta} = e^{i(\frac{3}{2}\theta + \frac{\pi}{2})}$$

so $\theta \in (-\frac{\pi}{3}, \frac{\pi}{3}) \rightsquigarrow \frac{3}{2}\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\rightsquigarrow \frac{3}{2}\theta + \frac{\pi}{2} \in (0, \pi)$.

- then we map from $\mathbb{H}$ to $\mathbb{D}$

$$\boxed{\eta = \frac{w - i}{w + i}}$$

$w = 0 \leftrightarrow \eta = -1$

$w = \infty \leftrightarrow \eta = 1$

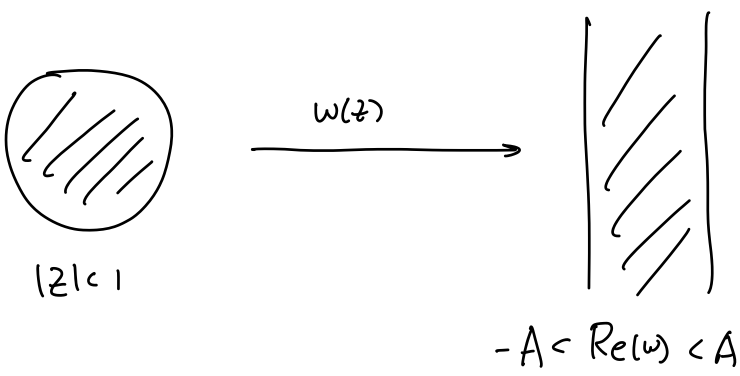
$w = i \leftrightarrow \eta = 0$.

Combine them. $\eta = \dfrac{i z^{3/2} - i}{i z^{3/2} + i} = \dfrac{z^{3/2} - 1}{z^{3/2} + 1}$.

These are not unique. we can choose any $r > 0$, and do

$$\eta = \frac{r \cdot z^{3/2} - 1}{r z^{3/2} + 1}$$

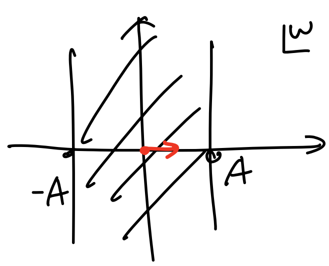
3. For fixed $A > 0$, find the conformal map $w(z)$ of the open unit disk $\{|z| < 1\}$ onto the vertical strip $\{-A < \operatorname{Re} w < A\}$ that satisfies $w(0) = 0$ and $w'(0) > 0$. Sketch the curves in the disk that correspond to vertical and horizontal lines in the strip.



Pf:

Consider the map from the strip to

, this maps the strip to the upper Half plane

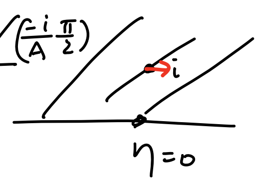


$$w = \log(-i\eta) / \left(\frac{-i}{A}\frac{\pi}{2}\right)$$

$$\eta(w) = i e^{-i \frac{w}{A} \frac{\pi}{2}}$$

$$\eta(i\infty) = i\infty$$

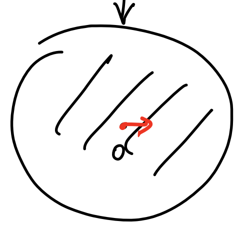
$$\eta(-i\infty) = 0$$



$\mathbb{H}$

$$\eta = -i\,\frac{z+1}{z-1}$$

$$z = \frac{\eta - i}{\eta + i}$$

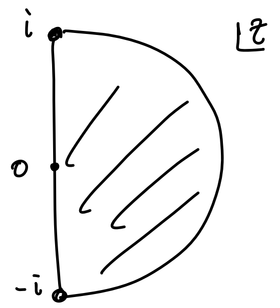


$\mathbb{D}$

z

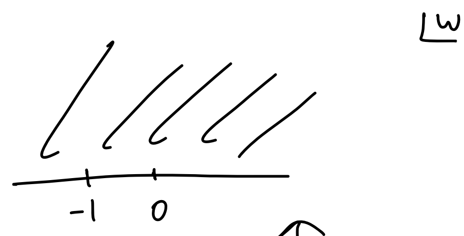
$$w = \frac{\log\left(\frac{1+z}{1-z}\right)}{\frac{-i}{A}\,\frac{\pi}{2}}$$

5. Find a conformal map $w(z)$ of the right half-disk $\{\operatorname{Re} z > 0,\ |z| < 1\}$ onto the upper half-plane that maps $-i$ to 0, $+i$ to ∞, and 0 to -1. What is $w(1)$?



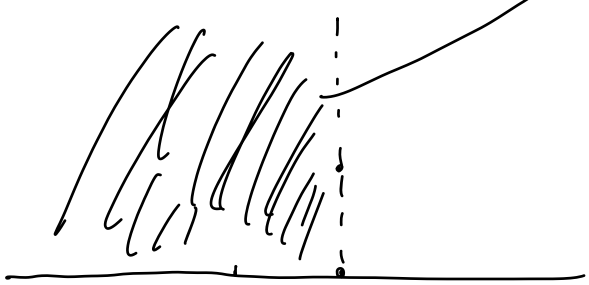
$\left\{ \begin{array}{l} |z|<1 \\ \operatorname{Re}(z)>0 \end{array} \right\}$

$0 \to -1$



$w = \left(\frac{\eta}{i}\right)^2$

$\eta = \frac{z+i}{z-i}$

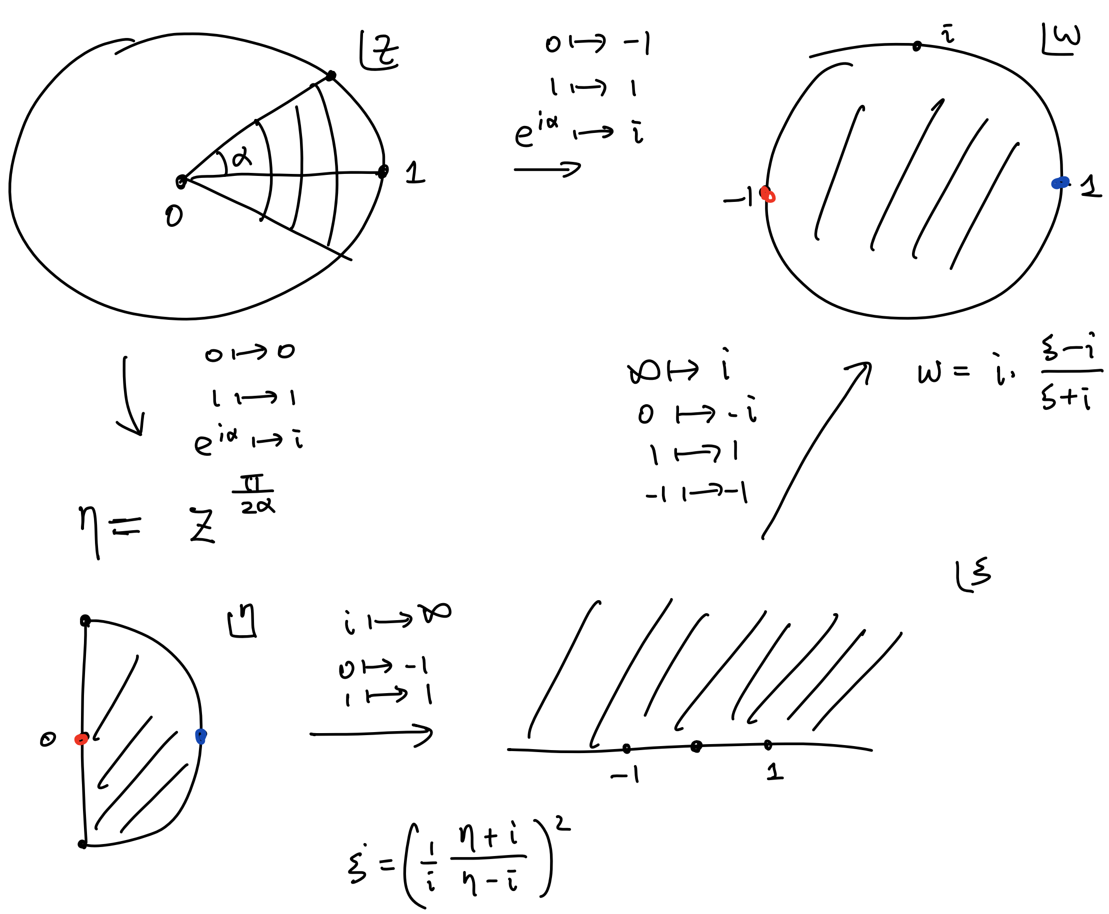


$\eta(1) = \frac{1+i}{1-i} = i$

$\eta(e^{i\theta}) = \frac{e^{i\theta}+i}{e^{-i\theta}-i} \in i\mathbb{R}$

$$w = \left(\frac{1}{i}\,\frac{z+i}{z-i}\right)^2$$

7. Find the conformal map of the pie-slice domain $\{|\arg z| < \alpha,\ |z| < 1\}$ onto the open unit disk such that $w(0) = -1$, $w(+1) = +1$, and $w(e^{i\alpha}) = i$. It is enough to express $w(z)$ as a composition of specific conformal maps.



$$\eta = z^{\frac{\pi}{2\alpha}}$$

$$\xi = \left(\frac{1}{i}\,\frac{\eta + i}{\eta - i}\right)^2$$

$$w = i \cdot \frac{\xi - i}{\xi + i}$$

8. For fixed b in the interval $(-1, 1)$, find all conformal maps of the unit disk slit along the interval $[-1, b]$ onto the entire unit disk that map b to -1 and leave $+1$ fixed. It is enough to express them as a composition of specific conformal maps.