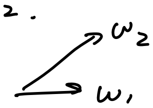
Stein Ch 10. #2, #4. + extra.

#2 Let f be an elliptic function with period ω_1, ω_2.

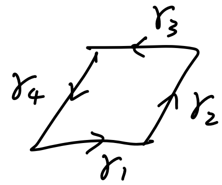


Let $a_1, \cdots, a_r$ be zeros &

$b_1, \cdots, b_r$ be poles in a fundamental parallelogram. P_0. Show that

$$a_1 + \cdots + a_r - (b_1 + \cdots + b_r) = n_1 w_1 + n_2 w_2$$

Pf: $\text{LHS} = \frac{1}{2\pi i} \oint_{\partial P_0} z \cdot \frac{f'(z)}{f(z)} \, dz$



$$= \frac{1}{2\pi i} \left[\int_{\gamma_1} z \frac{f'(z)}{f(z)} \, dz + \int_{\gamma_3} z \cdot \frac{f'(z)}{f(z)} \, dz + \int_{\gamma_2} z \frac{f'(z)}{f(z)} \, dz + \int_{\gamma_4} z \cdot \frac{f'(z)}{f(z)} \, dz \right]$$

$$= \frac{1}{2\pi i} \left[\int_{\gamma_1} z \frac{f'(z)}{f(z)} \, dz - \int_{\gamma_1} (z+w_2) \frac{f'(z+w_2)}{f(z+w_2)} \, d(z+w_2) + \int_{\gamma_2} z \frac{f'(z)}{f(z)} \, dz - \int_{\gamma_2} (z-w_1) \frac{f'(z-w_1)}{f(z-w_1)} \, d(z-w_1) \right]$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} (-w_2) \cdot \frac{f'(z)}{f(z)} \, dz$$

$$+ \frac{1}{2\pi i} \int_{\gamma_2} (w_1) \frac{f'(z)}{f(z)} \, dz.$$

observe that $\displaystyle\int_{\gamma_i} \frac{f'(z)}{f(z)} \, dz = \int_{f(\gamma_i)} \frac{dw}{w} = n \cdot 2\pi i$ for some $n \in \mathbb{Z}$.

where $f(\gamma_i)$ is a closed loop in $\mathbb{C}$, away from 0

Thus, LHS $= n_1 w_1 + n_2 w_2$, $n_i \in \mathbb{Z}$.

<u>#4</u> Prove that $\displaystyle P(z) = \frac{1}{z^2} + \sum_{w \in \Lambda^*} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right]$

is doubly periodic without using differentiation.

<u>Pf</u>: define $\displaystyle P^R(z) = \frac{1}{z^2} + \sum_{\substack{w \in \Lambda^* \\ |w| < R}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right]$

Then. $\displaystyle P(z) - P^R(z) = \sum_{\substack{w \in \Lambda^* \\ |w| > R}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right]$

We claim that, for $R > 2|z|$, we have

$$\sum_{\substack{w \in \Lambda^* \\ |w| > R}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \leq \frac{C}{R}$$

for some constant C. that only depends on Λ.

(Proof of claim)

Indeed, $\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = \left| \frac{-z^2+2zw}{(z-w)^2 w^2} \right| = O\left(\frac{1}{|w|^3}\right)$ as $|w| \to \infty$

and $\displaystyle\sum_{\substack{w \in \Lambda^* \\ |w|>R}} \frac{1}{|w|^3} = \sum_{\substack{|m+n\tau|>R \\ m,n \in \mathbb{Z}}} \frac{1}{|m+n\tau|^3} \leq \sum_{(m,n) \in \mathbb{Z}^2 \setminus [-R/2, R/2]^2} \frac{1}{|m+n\tau|^3}$

By Lemma 1.5 in Stein, $\exists\, c>0$, s.t. $|m+n\tau| > \frac{1}{c}(|n|+|m|)$

$$\therefore \quad \sum \frac{1}{|m+n\tau|^3} \leq \sum \frac{1}{\frac{1}{c^3}(|m|+|n|)^3}$$

$$\leq c^3 \iint_{(x,y) \in \mathbb{R}^2 \setminus [-\frac{R}{2}, \frac{R}{2}]^2} \frac{1}{(|x|+|y|)^3}\, dx\,dy \leq C_1 \cdot \iint_{|(x,y)|>\frac{R}{2}} \frac{1}{(|(x,y)|)^3}\, dx\,dy$$

$$\leq C_2 \iint_{r>\frac{R}{2}} \frac{1}{r^3}\, r\,dr\,d\theta \leq C_3 \cdot \int_{r=\frac{R}{2}}^{\infty} \frac{1}{r^2}\, dr$$

$$\leq C_4 \cdot \frac{1}{R}.$$

where C_i are suitable constants independent of R.

(end of proof of claim).

Also, we note that

$$P^R(z) - P^R(z+1) = O\left(\frac{1}{R}\right)$$

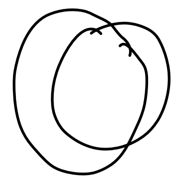
Pf: Let $\Lambda_R = \{w \in \Lambda \mid |w| < R\}$.

let $\Lambda_R + c = \{ w + c \mid w \in \Lambda_R \}$.

then $p^R(z) - p^R(z+1)$

$$= \sum_{w \in \Lambda_R} \frac{1}{(z-w)^2} - \sum_{w \in \Lambda_R - 1} \frac{1}{(z-w)^2}$$

$$= \sum_{\substack{w \in \Lambda_R \\ w \notin \Lambda_R - 1}} \frac{1}{(z-w)^2} - \sum_{\substack{w \in \Lambda_R - 1 \\ w \notin \Lambda_R}} \frac{1}{(z-w)^2}$$



$\therefore$ $\exists c$, say $c = 2$, such that

$$| p^R(z) - p^R(z+1) | \leq \sum_{\substack{R-c < |w| < R+c \\ w \in \Lambda}} \frac{1}{|z-w|^2} = O\left(\frac{1}{R}\right)$$

Hence, $P(z) - P(z+1) = p^R(z) - p^R(z+1) + O\left(\frac{1}{R}\right)$

$$= O\left(\frac{1}{R}\right)$$

as $R \to \infty$, we see $P(z) = P(z+1)$

similarly $P(z) - P(z+\tau) = O\left(\frac{1}{R}\right)$

hence as $R \to \infty$, $P(z) = P(z+\tau)$.

#

#3 (1) use series to construct an elliptic function with poles at $p_1, p_2 \in \{ (a + b\tau) \mid a, b \in [0,1) \}$.

assuming $p_1, p_2 \notin \Lambda$.

let $f(z) = \dfrac{1}{(z-p_1)(z-p_2)}$

let $F(z) = \sum_{w \in \Lambda} \left[f(z+w) - f(w) \right]$

then we may check that $F(z)$ is well defined.

$$F(z+1) = F(z), \quad F(z+\tau) = F(z).$$ (say, using problem #2 above).

(2) By the thm in Stein, we certainly can express any elliptic function using P and P'.

Here is another way,

Let $p_0 = (p_1+p_2)/2$, then

$G(z) = F(z+p_0)$ has simple poles at $\pm \frac{p_1-p_2}{2} = \pm z_0$.

$(2z_0 = p_1 - p_2 \notin \Lambda)$

Then. consider the function

$H(z) = P(z) - P(z_0)$. it has simple zeros at $\pm z_0$,

thus $G(z)\cdot H(z) = \text{const} = C$

$$\therefore \quad F(z) = G(z-p_0) = \frac{C}{H(z-p_0)} = \frac{C}{P(z-\frac{p_1+p_2}{2}) - P(\frac{p_1-p_2}{2})}$$

(3) The function f is not unique:

$c_1\cdot f + c_2$ for any $c_1 \in \mathbb{C}\setminus 0$, $c_2 \in \mathbb{C}$

also have the same poles.