Math 185: Homework 3 Solution

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The following exercises are from Stein's textbook, Chapter 1, prob 25, and Ch2: 1, 2,3,4

Problem (1.25). *The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.*

(a) Evaluate the integrals

$$\int_\gamma z^n dz$$

where $n \in$ and γ is any circle centers at the origin with the positive clockwise orientation

(b) Same question as before, but with γ any circle not containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Solution. For (a), we can parameterize $z = re^{i\theta}$ for θ running from 0 to 2π. Then

$$\int_\gamma z^n dz = \int_0^{2\pi} r^n e^{in\theta} re^{i\theta} i d\theta = r^{1+n} i \int_0^{2\pi} e^{i(1+n)\theta} d\theta = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{else} \end{cases}$$

Alternatively, for $n \neq -1$, we can find primitive of z^n as $z^{n+1}/(n+1)$ over $\mathbb{C}\backslash\{0\}$, then one can apply Corollary 3.2.

For (b), we parameterize the circle as $z = z_0 + re^{i\theta}$ with $|z_0| > r$. Then again over the cirlce, for $n \neq -1$ we can find primitive of z^n, hence the integral is zero. Suffice to consider the case $n = -1$, thus we have

$$\int_\gamma z^{-1} dz = \int_0^{2\pi} \frac{re^{i\theta}}{z_0 + re^{i\theta}} i d\theta$$

Since $|z_0| >$, we can expand the integrand

$$\frac{re^{i\theta}}{z_0 + re^{i\theta}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{re^{i\theta}}{z_0}\right)^{n+1}$$

Now we are going to switch the order of summation and integration, again, we check the absolute convergence, namely

$$\int_0^{2\pi} \sum_{n=0}^{\infty} \left(\frac{r}{|z_0|} \right)^{n+1} d\theta = 2\pi \frac{r}{|z_0|} \frac{1}{1 - \frac{r}{|z_0|}} < \infty.$$

Hence

$$\int_0^{2\pi} \frac{re^{i\theta}}{z_0 + re^{i\theta}} id\theta = \sum_{n=0} (-1)^n \int_0^{2\pi} \left(\frac{re^{i\theta}}{z_0} \right)^{n+1} d\theta = 0$$

Finally, for (c). We can write

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left(\frac{1}{(z-a)} - \frac{1}{(z-b)} \right),$$

and do integration for both terms. The first term is like (a) where the point a is within the circle $|z| = r$, the second term is like (b) and contribution is zero.

The integral for the first term can be computered using power series again

$$\int_{|z|=r} \frac{1}{z-a} dz = \int_{|z|=r} \frac{1}{z(1-a/z)} dz = \int_{|z|=r} z^{-1}(1 + a/z + (a/z)^2 + \cdots) dz$$

$$= \sum_{n=0}^{\infty} \int_{|z|=r} z^{-1}(a/z)^n dz = \int_{|z|=r} z^{-1} dz = 2\pi i$$

where when we switch the summation and integral, we again checked that the double sum (more precisely, the integral-sum, is absolutely convergent, meaning if we take the absolute value of the summand-integrand, the integral is still finite).

Alternatively, you can use the Cauchy integral formula in Ch2 to do this problem.