Math 185: Homework 3 Solution

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The following exercises are from Stein's textbook, Chapter 1, prob 25, and Ch2: 1, 2,3,4

Problem (1.25)**.** *The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.*

(a) Evaluate the integrals

$$\int_\gamma z^n dz$$

where $n \in$ *and* γ *is any circle centers at the origin with the positive clockwise orientation*

(b) Same question as before, but with γ *any circle not containing the origin.*

(c) Show that if $|a| < r < |b|$*, then*

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ *denotes the circle centered at the origin, of radius* r*, with the positive orientation.*

Solution. For (a), we can parameterize $z = re^{i\theta}$ for θ running from 0 to 2π. Then

$$\int_\gamma z^n dz = \int_0^{2\pi} r^n e^{in\theta} re^{i\theta} i d\theta = r^{1+n} i \int_0^{2\pi} e^{i(1+n)\theta} d\theta = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{else} \end{cases}$$

Alternatively, for $n \neq -1$, we can find primitive of z^n as $z^{n+1}/(n+1)$ over $\mathbb{C}\backslash\{0\}$, then one can apply Corollary 3.2.

For (b), we parameterize the circle as $z = z_0 + re^{i\theta}$ with $|z_0| > r$. Then again over the cirlce, for $n \neq -1$ we can find primitive of z^n, hence the integral is zero. Suffice to consider the case $n = -1$, thus we have

$$\int_\gamma z^{-1} dz = \int_0^{2\pi} \frac{re^{i\theta}}{z_0 + re^{i\theta}} i d\theta$$

Since $|z_0| >$, we can expand the integrand

$$\frac{re^{i\theta}}{z_0 + re^{i\theta}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{re^{i\theta}}{z_0} \right)^{n+1}$$

Now we are going to switch the order of summation and integration, again, we check the absolute convergence, namely

$$\int_0^{2\pi} \sum_{n=0}^{\infty} \left(\frac{r}{|z_0|}\right)^{n+1} d\theta = 2\pi \frac{r}{|z_0|} \frac{1}{1-\frac{r}{|z_0|}} < \infty.$$

Hence

$$\int_0^{2\pi} \frac{re^{i\theta}}{z_0 + re^{i\theta}} id\theta = \sum_{n=0} (-1)^n \int_0^{2\pi} \left(\frac{re^{i\theta}}{z_0}\right)^{n+1} d\theta = 0$$

Finally, for (c). We can write

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{(z-a)} - \frac{1}{(z-b)}\right),$$

and do integration for both terms. The first term is like (a) where the point a is within the circle $|z| = r$, the second term is like (b) and contribution is zero.

The integral for the first term can be computered using power series again

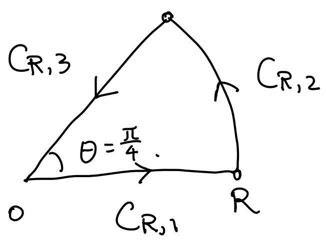
$$\int_{|z|=r} \frac{1}{z-a} dz = \int_{|z|=r} \frac{1}{z(1-a/z)} dz = \int_{|z|=r} \frac{1}{z(1-a/z)} = \int_{|z|=r} z^{-1}(1+a/z+(a/z)^2+\cdots)dz$$

$$= \sum_{n=0}^{\infty} \int_{|z|=r} z^{-1}(a/z)^{-n} dz = \int_{|z|=r} z^{-1} dz = 2\pi i$$

where when we switch the summation and integral, we again checked that the double sum (more precisely, the integral-sum, is absolutely convergent, meaning if we take the absolute value of the summand-integrand, the integral is still finite).

#1. Prove that $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$

Proof: Consider the contour integral $\int_{C_R} e^{-z^2} dz$, where C_R is as follows



- Since e^{-z^2} is an entire function, by Cauchy theorem, we have $\int_{C_R} e^{-z^2} dz = 0$.

- $I_1 = \int_{C_{R,1}} e^{-z^2} dz = \int_0^R e^{-x^2} dx$

$$I_2 = \int_{C_{R,2}} e^{-z^2}\, dz = \int_{\theta=0}^{\pi/4} e^{-(Re^{i\theta})^2} \cdot d(Re^{i\theta})$$

$$= \int_{\theta=0}^{\pi/4} e^{-R^2 e^{2i\theta}} Re^{i\theta} \cdot i\, d\theta$$

$$I_3 = \int_{C_{R,3}} e^{-z^2}\, dz = \int_{r=R}^{0} e^{-(re^{i\frac{\pi}{4}})^2}\, d(r \cdot e^{i\frac{\pi}{4}})$$

$$= -e^{i\frac{\pi}{4}} \int_0^R e^{-ir^2}\, dr = -e^{i\frac{\pi}{4}} \cdot \int_0^R \cos(r^2) - i\sin(r^2)\, dr$$

We claim that : $I_2 \to 0$ as $R \to \infty$. Given the claim, we have

$$\int_0^\infty \cos(r^2) - i \cdot \sin(r^2)\, dr = \lim_{R\to\infty} \left(\frac{I_3}{-e^{i\pi/4}}\right) = \lim_{R\to\infty} \frac{-I_1 - I_2}{-e^{i\pi/4}}$$

$$= \lim_{R\to\infty} \frac{I_1}{e^{i\pi/4}} = e^{-i\frac{\pi}{4}} \cdot \frac{\sqrt{\pi}}{2} = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}.$$

Since $\int_0^\infty \cos r^2\, dr$ and $\int_0^\infty \sin(r^2)\, dr$ are both real, we can compare the real and imaginary parts of the above equation and get the desired results.

Now, we turn back to prove the claim.

$$|I_2| = \left| \int_{\theta=0}^{\pi/4} e^{-R^2 e^{2i\theta}} Re^{i\theta} i \cdot d\theta \right|$$

$$\leq \int_0^{\pi/4} e^{-R^2 \cos 2\theta} R \cdot d\theta$$

For $\theta \in [0, \pi/4]$, let $\theta = \frac{\pi}{4} - u$, we have

$$\cos(2\theta) = \cos\left(\frac{\pi}{2} - 2u\right) = \sin(2u) \geq \frac{4}{\pi} u \quad \text{for } u \in [0, \tfrac{\pi}{4}].$$

$$\int_0^{\frac{\pi}{4}} e^{-R^2\cos 2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} e^{-R^2 \sin(2u)}\, du \leq \int_0^{\frac{\pi}{4}} e^{-R^2\cdot\frac{4}{\pi}u}\, du$$

$$\leq \int_0^{\infty} e^{-R^2\frac{4}{\pi}\cdot u}\, du = \frac{\pi}{4R^2}.$$

Thus. $|I_2| \leq R\cdot\frac{\pi}{4R^2} = \frac{\pi}{4}\cdot\frac{1}{R} \longrightarrow 0$ as $R\to\infty$. #.

#2 Show that $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$

Proof : $$\int_0^{\infty}\frac{\sin x}{x}\,dx = \lim_{\substack{\varepsilon\to 0,\\ R\to\infty}} \int_\varepsilon^R \frac{\sin x}{x}\,dx = \lim_{\cdots}\int_\varepsilon^R \frac{e^{ix}-e^{-ix}}{2ix}\,dx$$

$$= \lim_{\cdots}\left(\int_\varepsilon^R \frac{e^{ix}}{2ix}\,dx + \int_\varepsilon^R \frac{-e^{-ix}}{2ix}\,dx\right)$$

let $x=-u$, then $u=-x$ from $-\varepsilon$ to $-R$

$$= \lim\left(\int_\varepsilon^R \frac{e^{ix}}{2ix}\,dx + \int_{-\varepsilon}^{-R}\frac{-e^{iu}}{-2iu}\,d(-u)\right)$$

switch direction of integration gives a minus sign.

$$= \lim\left(\int_\varepsilon^R \frac{e^{ix}}{2ix}\,dx + \int_{-R}^{-\varepsilon}\frac{e^{iu}}{2iu}\,du\right)$$

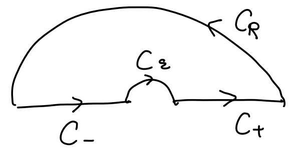
$$= \lim\left(\int_{-R}^{-\varepsilon} + \int_\varepsilon^R\right)\frac{e^{ix}}{2ix}\,dx \quad \text{------} \quad (*)$$

Note that, $\displaystyle\int_\varepsilon^R \frac{1}{x}\,dx = -\int_{-R}^{-\varepsilon}\frac{1}{x}\,dx.$, hence the above integral is also

$$\lim_{\cdots}\left(\int_{-R}^{-\varepsilon} + \int_\varepsilon^R\right)\frac{e^{ix}-1}{2ix}\,dx. \quad \cdots\cdots \quad (**).$$

as suggested by the hint.

In the following, I will give 2 solutions using $(*)$ or. $(**)$.

Using $(*)$: Consider the contour 

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Then
$$\int_{C_\varepsilon} \frac{e^{iz}}{2iz}\,dz = \int_{C_\varepsilon} \frac{e^{iz}-1}{2iz} + \frac{1}{2iz}\,dz$$

since $(e^{iz}-1)/(2iz)$ is a bounded function near $z=0$.,

$\int_{C_\varepsilon} \frac{e^{iz}-1}{2iz}\,dz \to 0$ as $\varepsilon \to 0$.

$$\int_{C_\varepsilon} \frac{1}{2iz}\,dz = \frac{1}{2i}\int_{\theta=\pi}^{0} \frac{1}{\varepsilon e^{i\theta}}\,\varepsilon\cdot e^{i\theta}\cdot i\,d\theta = \frac{1}{2i}(-\pi i) = -\frac{\pi}{2}.$$

$$\int_{C_R} \frac{e^{iz}}{2iz}\,dz \to 0 \quad \text{as } R\to\infty \text{ by Jordan Lemma.}$$

Thus
$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{C_-} + \int_{C_+}\right) \frac{e^{iz}}{2iz}\,dz = -\lim_{\cdots}\left(\int_{C_\varepsilon} + \int_{C_R}\right)\frac{e^{iz}}{2iz}\,dz = \frac{\pi}{2}.$$

Using $(**)$: Consider the same contour, now we have.

$$\int_{C_\varepsilon} \frac{e^{iz}-1}{2iz}\,dz \to 0 \quad \text{as } \varepsilon\to 0$$

$$\int_{C_R} \frac{e^{iz}-1}{2iz}\,dz = \int_{C_R} \frac{-1}{2iz}\,dz + \int_{C_R} \frac{e^{iz}}{2iz}\,dz$$

$$\to \int_{C_R} \frac{-1}{2iz}\,dz \qquad \text{as } R\to\infty$$

$$= \frac{-1}{2i}\cdot(\pi i) = -\frac{\pi}{2}.$$

Thus
$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{C_-} + \int_{C_+}\right) \frac{e^{iz}-1}{2iz}\,dz = -\lim_{\cdots}\left(\int_{C_\varepsilon} + \int_{C_R}\right)\frac{e^{iz}-1}{2iz}\,dz = \frac{\pi}{2}.$$

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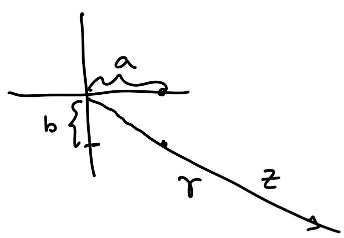
#3 Evaluate the integral

$$\int_0^\infty e^{-ax} \cos(bx)\, dx \qquad \int_0^\infty e^{-ax} \cdot \sin(bx)\, dx, \quad a>0.$$

Sol'n: $\displaystyle\int_0^\infty e^{-ax} \cos(bx)\, dx = \mathrm{Re}\left(\int_0^\infty e^{-ax} \cdot e^{ibx}\, dx\right)$

Let $z = (a-ib)x$, then as x goes from 0 to ∞

z goes along $(a-ib)\cdot \mathbb{R}_{\geq 0}$ ray (call it γ) to ∞. $\quad \dfrac{dz}{dx} = a-ib \quad \therefore\ dx = \dfrac{dz}{a-ib}$



$$I = \int_0^\infty e^{-ax+ibx}\, dx = \int_\gamma e^{-z} \frac{dz}{a-ib} = \frac{a+ib}{a^2+b^2} \int_\gamma e^{-z}\, dz$$

$$= \frac{a+ib}{a^2+b^2} \left(\frac{e^{-z}}{-1}\right)_0^\infty = \frac{a+ib}{a^2+b^2}$$

$$\int_0^\infty e^{-ax} \sin bx\, dx = \mathrm{Re}(I) = \frac{a}{a^2+b^2}$$

$$\int_0^\infty e^{-ax} \sin bx\, dx = \mathrm{Im}(I) = \frac{b}{a^2+b^2}$$

#4 Prove that for all $\xi \in \mathbb{C}$, we have

$$e^{-\pi\cdot\xi^2} = \int_{-\infty}^{+\infty} e^{-\pi x^2}\cdot e^{2\pi i\cdot x\cdot \xi}\, dx.$$

Pf: Consider the RHS of the equation:

$$\int_{-\infty}^{+\infty} e^{-\pi(x^2-2ix\xi)}\, dx = \int_{-\infty}^{+\infty} e^{-\pi\left((x-i\xi)^2-(i\xi)^2\right)}\, dx$$

$$= \int_{-\infty}^{+\infty} e^{-\pi(x-i\xi)^2-\pi\xi^2}\, dx.$$

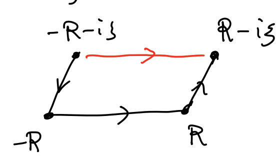
Hence, suffice to prove that

$$\int_{-\infty}^{+\infty} e^{-\pi(x-i\xi)^2}\, dx = 1.$$

Now $\lim\limits_{R\to\infty} \int_{-R}^{R} e^{-\pi(x-i\xi)^2}\, dx = \lim\limits_{R\to\infty} \int_{-R-i\xi}^{R-i\xi} e^{-\pi u^2}\, du.$

where $u = x-i\xi$.

$$= \lim_{R\to\infty}\left(\int_{-R-i\xi}^{-R} + \int_{-R}^{R} + \int_{R}^{R-i\xi}\right) e^{-\pi u^2}\, du$$



Suffice to prove that $\int_{-R-i\xi}^{-R} e^{-\pi u^2}\, du \to 0$ and $\int_{R}^{R-i\xi} e^{-\pi u^2}\, du \to 0.$

$$\left|\int_{R}^{R-i\xi} e^{-\pi u^2}\, du\right| = \left|\int_{t=0}^{1} e^{-\pi(R-i\xi t)^2}\, d(R-i\xi t)\right|$$

$$|e^z| = e^{\operatorname{Re}(z)}$$

$$\le \int_0^1 e^{-\pi\cdot\operatorname{Re}(R-i\xi t)^2}\, |\xi|\cdot dt$$

$$\le \left(\max_{t\in[0,1]} e^{-\pi\cdot\operatorname{Re}((R-i\xi t)^2)}\right)\cdot|\xi|$$

$\because \quad \mathrm{Re}\left((R - i\xi t)^2 \right) = \mathrm{Re}\left(R^2 - 2i\xi t \cdot R + (i\xi t)^2 \right)$

$$\geq R^2 - R \cdot 2|\xi| \cdot t - |\xi|^2 t^2$$

$$\geq R^2 - 2|\xi| R - |\xi|^2$$

Hence, as $R \to \infty$, $\mathrm{Re}\left((R - i\xi t)^2 \right) \to \infty$ uniformly in $t \in [0,1]$.

$\therefore \quad \left| \int_R^{R-i\xi} e^{-\pi u^2} \, du \right| \to 0$ as $R \to \infty$.