Stein Ch 3 # 9, 12, 16. + 2 extra problems.

#9. $\int_0^1 \log(\sin(\pi x))\,dx = -\log 2.$

Pf : • First, we check that the integral is well defined near $x=0$ and 1, where $\sin(x) \to 0^+$. Indeed,

$$\int_0^1 \log x \, dx = \int_\infty^1 \log\left(\frac{1}{u}\right) d\left(\frac{1}{u}\right) = -\int_1^\infty \frac{\log u}{u^2}\, du$$

which is convergent, since $\log u < C \cdot u^{\frac{1}{2}}$ for some $C>0$ for any $u>1$.

and $\int_1^\infty \frac{u^{\frac{1}{2}}}{u^2}\, du < \infty$

$$I = \lim_{\varepsilon\to 0} \int_\varepsilon^{1-\varepsilon} \log(\sin(\pi x))\,dx$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} \log\left(\frac{e^{\pi i x} - e^{-\pi i x}}{2i}\right) dx$$

For $x \in (\varepsilon, 1-\varepsilon)$, we may choose arg, s.t.

numerator $\arg\left(e^{\pi i x} - e^{-\pi i x}\right) = \frac{\pi}{2}$

denominator. $\arg(i) = \frac{\pi}{2}$.

Thus

$$\log\left(\frac{e^{\pi i x} - e^{-\pi i x}}{2i}\right) = \log\left(e^{\pi i x} - e^{-\pi i x}\right) - \log(2i)$$

with the above choice of arg understood. Further more

$$\log\left(e^{\pi i x} - e^{-\pi i x}\right) = \log e^{-\pi i x}\left(e^{2\pi i x} - 1\right)$$

$$= \log e^{-\pi i x} + \log\left(e^{2\pi i x} - 1\right)$$

with $\arg\left(e^{-\pi i x}\right) \in (0, -\pi)$

and $\arg\left(e^{2\pi i x} - 1\right) \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$

when x varies in $(\varepsilon, 1-\varepsilon)$.

$$= -\pi i x + \log\left(e^{2\pi i x} - 1\right).$$

$$\int_0^1 -\log(2i)\, dx = \int_0^1 -\left[\log 2 + \frac{\pi}{2} i\right] dx = -\left[\log 2 + \frac{\pi i}{2}\right]$$

$$\int_0^1 -\pi i x \quad dx = -\pi i \cdot \frac{x^2}{2}\Big|_0^1 = -\frac{\pi i}{2}$$

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1-\varepsilon} \log\left(e^{2\pi i x} - 1\right) dx$$

$$= \lim_{\varepsilon \to 0} \int_{\substack{z = e^{i 2\pi \cdot x} \\ x \in (\varepsilon, 1-\varepsilon)}} \log(z-1) \frac{dz}{2\pi i \cdot z}$$

$$= \lim_{\varepsilon \to 0} \int \log(z-1) \frac{dz}{2\pi i \cdot z}$$

• the added small arc's contribution $\to 0$ as $\varepsilon \to 0$. by the estimate in the beginning.

$$= \log(-1) = \pi i$$

$\because$ arg $(z-1)$ in this contour was chosen to be within $\left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$.

Put these terms together, we get

$$I = -\left(\log 2 + \frac{\pi}{2} i\right) - \frac{\pi}{2} i + \pi i = -\log 2.$$

12.

12. Suppose u is not an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}$$

by integrating

$$f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$$

over the circle $|z| = R_N = N + 1/2$ (N integral, $N \geq |u|$), adding the residues of f inside the circle, and letting N tend to infinity.
Note. Two other derivations of this identity, using Fourier series, were given in Book I.

Let $$I_N = \int_{|z|=R_N} \frac{\pi \cot(z)}{(u+z)^2} \frac{dz}{2\pi i}$$

$$f(z) = \frac{\pi \cot(\pi z)}{(u+z)^2} = \frac{1}{(u+z)^2} \frac{\cos(\pi z)}{\sin(\pi z)}$$

It has simple poles at $z \in \{-N, -N+1, \cdots, N\}$.
a double pole at $z = -u$.

$$\operatorname{Res}_{z=n} f(z) = \frac{\pi}{(u+z)^2} \left. \frac{\cos(n\pi)}{\sin(z)'} \right|_{z=n} = \frac{1}{(u+n)^2}$$

$$\operatorname{Res}_{z=u} f(z) = \frac{\pi}{(u+z)^2} \left[\cot(\pi u) + (u+z)\frac{-\pi}{\sin^2(\pi u)} + \cdots \right]$$

coeff of $(u+z)^{-1}$

$$\left(\cot'(x) = -\frac{1}{\sin^2(x)} \right) \qquad = \frac{-\pi^2}{\sin^2(\pi u)}$$

Hence $$I_N = \sum_{n=-N}^{N} \frac{1}{(u+n)^2} + \frac{-\pi^2}{\sin^2(\pi u)}.$$

Now, it suffices to prove that $\lim_{N\to\infty} I_N \to 0$.

We just need to show $\cot(\pi z)$ is bounded, when

$|z| = R_N = N + \frac{1}{2}$. $\quad \cot(\pi z) = \dfrac{e^{2\pi i z}+1}{e^{2\pi i z}-1} \cdot i$

For $y > 1$, $\quad \left|1 - e^{2\pi i (x+iy)}\right| \geqslant 1 - e^{-2\pi y}$

$$\geqslant 1 - e^{-2\pi}$$

For $0<y<1$. $\quad \left|1 - e^{2\pi i (x+iy)}\right|$

$$\begin{aligned}
&\geqslant \operatorname{Re}\left(1 - e^{2\pi i (x+iy)}\right) \\
&= 1 - e^{-2\pi y} \cos(2\pi x) \\
&= 1 - e^{-2\pi y} \cos(2\pi (x-R) + 2\pi R) \\
&= 1 + e^{-2\pi y} \cos\left(2\pi \frac{x^2 - R^2}{x+R}\right) \\
&= \underbrace{1 + e^{-2\pi y} \cos\left(2\pi \frac{-y^2}{x+R}\right)}
\end{aligned}$$

as $\lim_{R_N \to 0}$, $\quad \longrightarrow 1 + e^{-2\pi y}$

we see $\quad \left|1 - e^{2\pi i (x+iy)}\right| > 1 \quad \forall\, 0<y<1$

for R_N sufficiently large.

similarly we can argue for $\operatorname{Im}(z) < 0$, using

$$\cot(\pi z) = i \, \frac{1 + e^{-2\pi i z}}{1 - e^{-2\pi i z}}$$

we get $\sup_{|z|=R_N} \left| \cot(\pi z) \right|$ is bounded from above by a constant C indep of N.

Finally:

$$\lim_{N\to\infty} \left| \oint \frac{1}{(u+z)^2} \cot(\pi z)\, dz \right|$$

$$\leq \lim_{R\to\infty} \frac{1}{R^2} \cdot 2\pi R \cdot M = 0$$

Thus $\lim_{N\to\infty} I_N = 0$.

#16

16. Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show that if ϵ is sufficiently small, then

(a) $f_\epsilon(z)$ has a unique zero in $|z| \leq 1$, and

(b) if z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

pf (a) follows from Rouché thm

(b). Suppose $\varepsilon_0 > 0$ is small enough, such that $\forall\, \varepsilon \in \mathbb{C}$, $0 \leq |\varepsilon| < \varepsilon_0$, $f_\varepsilon(z)$ has unique zero z_ε inside $|z| \leq 1$. and $|z_\varepsilon| < 1$. Then.,

$$z_\varepsilon = \frac{1}{2\pi i} \int_C \frac{f_\varepsilon'(z)}{f_\varepsilon(z)}\, z\, dz$$

indeed. $\dfrac{f_\varepsilon'(z)}{f_\varepsilon(z)} = \left(\dfrac{1}{z - z_\varepsilon} + \text{hol'c part} \right)$.

Hence $\dfrac{1}{2\pi i} \displaystyle\int_C \left(\frac{1}{z - z_\varepsilon} + \text{hol'c} \right) \cdot z\, dz = z_\varepsilon$.

Since the integrand $\dfrac{f_\varepsilon'(z)}{f_\varepsilon(z)} \cdot z$ is continuous in ε., for $|\varepsilon| < \varepsilon_0$, and domain C is compact, hence the integrand is uniformly continuous in ε. Thus the result of the integral is continuous in ε.

actually, the integrand is hol'c in ε. and z_ε is hol'c in ε for $|\varepsilon| < \varepsilon_0$

4. How many roots does $z^4 - 6z + 3 = 0$ have on the annulus $1 < |z| < 2$?

Ans: we will count roots inside $|z|<2$ and $|z|<1$.

For $|z|< 2$, we let $f(z) = z^4$, $g(z) = -6z+3$. Then on $|z|=2$, $|f(z)|= 16$, $|g(z)| \leq |-6z| + 3 = 15$. , hence.

$f(z) + g(z)$ has the same number of roots as $f(z)$ inside $|z|=2$, namely 4.

For $|z|<1$, let $f(z) = -6z$, $g(z) = z^4+3$, then on $|z|$, $|f(z)| = 6$, $|g(z)| \leq 4$. we have number of roots of $f+g$ equal to that of f inside $|z|=1$, namely 1.

Hence, # of roots of $z^4 - 6z + 1$ has $4-1 = 3$ roots in the annulus $1<|z|<2$.

#5: Are the following open maps ?

1) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$.

yes, since complex conjugation is a linear isomorphism hence is open.

2) $f: \mathbb{C} \to \mathbb{R}$, $f(z) = |z|^2$.

Not open. for any open ball $B_r(0)$,

$f(\breve{B_r(0)}) = [0, r^2)$, which is not open.

3) $f: \mathbb{C} \to \mathbb{R}$. $\quad f(x+iy) = x \cdot y$.

Yes. $f(z) = \operatorname{Im}(z^2/2)$, hence f is a composition of a holomorphic map $\mathbb{C} \to \mathbb{C}$, with a projection $\mathbb{C} \to \mathbb{R}$, both are open map, hence f is open

4) $f: \mathbb{C} \to \mathbb{R}$ $\quad f(z) = \operatorname{Re}(z^3 + 2z)$.

open. same reason as 3).