Stein Ch 4. 3. 4. 6. 8. 9(a)

#3 Show by contour integral, that if $a>0$, $\xi \in \mathbb{R}$, we have

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi}\, dx = e^{-2\pi a|\xi|}$$

and that

$$\int_{-\infty}^{+\infty} e^{-2\pi a|\xi|} e^{2\pi i x \xi}\, d\xi = \frac{1}{\pi}\frac{a^2}{a^2+x^2}$$

Sol'n : (a) if $\xi \geq 0$, we can complete the contour by

since $\operatorname{Re}(-2\pi i \cdot z \cdot \xi) = \operatorname{Im}(2\pi z \xi) = 2\pi \xi \operatorname{Im}(z) \to -\infty$

as $\operatorname{Im}(z) \to -\infty$.

we get $$\oint \frac{a}{\pi(a^2+z^2)} e^{-2\pi i \cdot z \cdot \xi} \cdot dz$$

$$= -2\pi i \cdot \operatorname{Re}_{z=-ai} \frac{a}{\pi(a^2+z^2)} e^{-2\pi i z \cdot \xi}$$

$$= -2\pi i \frac{a}{\pi(-2ai)} \cdot e^{-2\pi a \xi} = e^{-2\pi a|\xi|}$$

Similarly, if $\xi \leq 0$, we may complete the contour.

$$\oint \frac{1}{\pi}\frac{a}{a^2+z^2} e^{-2\pi i \cdot z \cdot \xi} dz = 2\pi i \cdot \operatorname{Res}_{z=ai} \frac{e^{-2\pi i \cdot z \cdot \xi} \cdot a}{(a^2+z^2)\pi}$$

$$= 2\pi i \cdot \frac{a}{\pi} \cdot \frac{e^{2\pi a \xi}}{2ai} = e^{-2\pi a|\xi|}$$

Hence, for all $\xi \in \mathbb{R}$, we get $e^{-2\pi a|\xi|}$.

(b) $$\int_{-\infty}^{+\infty} e^{-2\pi a|\xi|} e^{2\pi i x\cdot\xi}\, d\xi$$

$$= \int_0^{\infty} e^{-2\pi a\xi + 2\pi i x\xi}\, d\xi + \int_{-\infty}^{0} e^{2\pi a\xi + 2\pi i\cdot x\xi}\, d\xi$$

$$= \int_0^{\infty} e^{-2\pi(a-ix)\xi}\, d\xi + \int_0^{\infty} e^{-2\pi(a+ix)\cdot\rho}\, d\rho$$

$$= \frac{1}{2\pi(a-ix)} + \frac{1}{2\pi(a+ix)} = \frac{1}{2\pi}\,\frac{(a+ix)+(a-ix)}{a^2+x^2}$$

$$= \frac{a}{\pi}\,\frac{1}{a^2+x^2}$$

#4: Suppose $Q$ is a polynomial of degree $\geq 2$, with distinct roots, none lying on the real axis. Calculate

$$I = \int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\xi}}{Q(x)}\, dx$$

in terms of the roots. What if the roots have multiplicities?

sol'n: Let $R_+$ denote the set of roots of $Q(x)$ with $\mathrm{Im} > 0$ and $R_-$ the set of roots with $\mathrm{Im} < 0$. $R = R_+ \cup R_-$

- If $\xi > 0$, we may complete the contour as

$$I = -2\pi i \sum_{a\in R_-} \operatorname{Res}_{z=a} \frac{e^{-2\pi i\cdot z\cdot\xi}}{Q(z)} = -2\pi i \sum_{a\in R_-} \frac{e^{-2\pi i\cdot a\cdot\xi}}{Q'(a)}$$

- If $\xi < 0$, then

$$I = 2\pi i \sum_{a\in R_+} \operatorname{Res}_{z=a} \frac{e^{-2\pi i z\xi}}{Q(z)} = 2\pi i \sum_{a\in R_+} \frac{e^{-2\pi i\cdot a\cdot\xi}}{Q'(a)}$$

- If $\xi = 0$, then we can use either contour, and get

$$I = 2\pi i \cdot \sum_{a \in R_+} \frac{1}{Q'(a)} = -2\pi i \sum_{a \in R_-} \frac{1}{Q'(a)}$$

Indeed
$$\sum_{a \in R_+ \cup R_-} \frac{1}{Q'(a)} = \oint_{|z| = R \gg 1} \frac{1}{Q(z)} \frac{dz}{2\pi i} = 0 .$$

Hence the above 2 expressions for $I$ agree.

- In above expressions,

$$Q(z) = C \cdot \prod_{a \in R} (z-a) \qquad \text{for some } C \in \mathbb{C}^*$$
$$Q'(a) = C \prod_{\substack{b \in R \\ b \neq a}} (a-b)$$

- Now, what if roots of $Q(x)$ coincides (this part is optional, and doesn't count into the grade)

Consider a simple example, say $\operatorname{Im}(a_i) > 0$

$$(*) \quad \oint_{-\infty}^{+\infty} \frac{e^{-2\pi i z \cdot \xi}}{(z-a_1)(z-a_2)} dz = 2\pi i \cdot \left( \frac{e^{-2\pi i \cdot a_1 \xi}}{a_1 - a_2} + \frac{e^{-2\pi i \cdot a_2 \cdot \xi}}{a_2 - a_1} \right)$$

$$(**) \quad \oint \frac{e^{-2\pi i z \cdot \xi}}{(z-a_1) \cdots (z-a_3)} dz = 2\pi i \cdot \left( \frac{e^{-2\pi i a_1 \xi}}{(a_1-a_2)(a_1-a_3)} + \frac{e^{-2\pi i \cdot a_2 \cdot \xi}}{(a_2-a_1)(a_2-a_3)} + \frac{e^{-2\pi i \cdot a_3 \cdot \xi}}{(a_3-a_1)(a_3-a_2)} \right)$$

For $(*)$, we may take limit $a_1 \to a$, $a_2 = a$, we get

$$\text{RHS} \to 2\pi i \cdot \left(e^{-2\pi i \cdot z \cdot \xi}\right)'\Big|_{z=a} = 2\pi i\,(-2\pi i\,\xi)\,e^{-2\pi i \cdot a\xi}$$

$$\text{LHS} \Rightarrow 2\pi i \operatorname{Res}_{z=a} \frac{e^{-2\pi i z\xi}}{(z-a)^2} = 2\pi i\,(-2\pi i\,\xi)\,e^{-2\pi i \cdot a\xi}$$

For (**).

its interesting to notice that, on the LHS, it converges to

$$2\pi i \cdot \frac{\left(e^{-2\pi i \cdot z \cdot \xi}\right)''}{2!}\Big|_{z=a}$$

on the RHS, we need to use some trick to see this.

say $a_i \to a$. let $f(z) = e^{-2\pi i \cdot z \cdot \xi}$.

Then $f(a_i) = f(a) + (a_i - a) f'(a) + \frac{1}{2!}(a_i - a)^2 f''(a) + \cdots$

we note

$$\sum_{i=1}^{3}\left(\prod_{j\neq i} \frac{1}{(a_i - a_j)}\right) = \oint \frac{z}{(z-a_1)(z-a_2)(z-a_3)} \frac{dz}{2\pi i} = 0$$

$$\sum_{i=1}^{3}\left(\prod_{j\neq i} \frac{1}{(a_i - a_j)}\right)\cdot a_i = \oint_{|z|=R\gg 1} \frac{z}{(z-a_1)(z-a_2)(z-a_3)} \frac{dz}{2\pi i} = 0$$

$$\sum_{i=1}^{3}\left(\prod_{j\neq i} \frac{1}{(a_i - a_j)}\right)\cdot a_i^2 = \oint_{|z|=R\gg 1} \frac{z^2}{(z-a_1)(z-a_2)(z-a_3)} \frac{dz}{2\pi i}$$

$$= \oint_{|z|} \left(\frac{1}{z} + \frac{\#}{z^2} + \frac{\#}{z^3} \cdots\right) \frac{dz}{2\pi i}$$

$$= 1$$

$$\text{Thus} \left(\lim_{a_i \to a} \sum_{i=1}^{3} \frac{f(a_i)}{\prod_{j\neq i}(a_i - a_j)}\right)$$

$$= \lim_{a_i \to a} \sum_{i=1}^{3} \frac{f(a) + (a_i - a) f'(a) + \frac{1}{2}(a_i - a)^2 f''(a) + O(|a_i - a|^3)}{\prod_{j \neq i} (a_i - a_j)}$$

$$= \lim_{a_i \to a} \left[ \frac{1}{2} f''(a) \right] + \sum_{i=1}^{3} \frac{O(|a_i - a|^3)}{\prod_{j \neq i} (a_i - a_j)}$$

$$= \frac{1}{2} f''(a).$$

#6 Prove that

$$\frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \frac{a}{a^2 + n^2} = \sum_{n=-\infty}^{+\infty} e^{-2\pi a |n|}$$

whenever $a > 0$

Pf : This follows from Poisson summation formula. (for $a>0$)

$$f(x) = \frac{1}{\pi} \frac{a^2}{a^2 + x^2} \xleftrightarrow{F.T} \hat{f}(\xi) = e^{-2\pi a |\xi|}$$

Now

$$\coth(\pi a) = \frac{e^{\pi a} + e^{-\pi a}}{e^{\pi a} - e^{-\pi a}} = \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}}$$

$$= (1 + e^{-2\pi a}) \left( \sum_{n=0}^{\infty} e^{-2\pi a \cdot n} \right)$$

$$= \sum_{n \in \mathbb{Z}} e^{-2\pi a |n|}$$

#8 Suppose $\hat{f}$ is supported in $[-M, M]$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Show that
$$a_n = \frac{(2\pi i)^n}{n!} \int_{-M}^{M} \hat{f}(\xi) \cdot \xi^n \, d\xi$$
and
$$\limsup_{n\to\infty} \left( n! \, |a_n| \right)^{\frac{1}{n}} \leq 2\pi M \qquad (**)$$

Conversely, if $(**)$ holds, show that $f(z)$ is hol'c in $\mathbb{C}$,

$\forall \varepsilon > 0$, $\exists A_\varepsilon > 0$. s.t.
$$|f(z)| \leq A_\varepsilon \, e^{2\pi (M+\varepsilon)|z|}$$

Pf : (1)
$$a_n = f^{(n)}(0) \cdot \frac{1}{n!}$$
$$= \frac{1}{n!} \left(\frac{d}{dz}\right)^n \int_{-M}^{M} \hat{f}(\xi) \, e^{2\pi i \cdot z \cdot \xi} \, d\xi$$
$$= \frac{1}{n!} \int_{-M}^{M} \hat{f}(\xi) \cdot \left(\frac{d}{dz}\right)^n e^{2\pi i z \cdot \xi} \, d\xi$$

$\because$ absolute convergence of $\int \left(\frac{d}{dz}\right)^n (\cdots)$

$$= \frac{1}{n!} \int_{-M}^{M} \hat{f}(\xi) \, (2\pi i \cdot \xi)^n \, d\xi$$
$$= \frac{(2\pi i)^n}{n!} \int_{-M}^{M} \hat{f}(\xi) \, \xi^n \, d\xi .$$

Hence $\limsup_{n\to\infty} (n!\,|a_n|)^{\frac{1}{n}}$

$$= \limsup_{n\to\infty} (2\pi)\cdot\left(\int_{-M}^{M} \hat{f}(\xi)\cdot \xi^n \, d\xi\right)^{\frac{1}{n}}$$

$$\leq \limsup_{n\to\infty} (2\pi)\cdot\left(\sup_{\xi\in[-M,M]} |\hat{f}(\xi)|\right)^{\frac{1}{n}} \cdot \left[\frac{M^{n+1}}{n+1} - \frac{(-M)^{n+1}}{n+1}\right]^{\frac{1}{n}}$$

$$= 2\pi\cdot M. \qquad \left(\because \lim_{n\to\infty} c^{\frac{1}{n}} = 1, \ \forall c>0.\right)$$

(b) For the converse direction:

$\forall \varepsilon>0$, let $N_\varepsilon > 0$ s.t. we have

$$(n!\,|a_n|)^{\frac{1}{n}} < 2\pi(M+\varepsilon) \qquad \forall n \geq N_\varepsilon,$$

Thus

$$\sum_{n=N_\varepsilon}^{\infty} |a_n|\cdot|z|^n \leq \sum_{n=N_\varepsilon}^{\infty} \frac{(2\pi(M+\varepsilon))^n}{n!}\cdot|z|^n$$

$$\leq e^{2\pi(M+\varepsilon)|z|}.$$

Now $\dfrac{\sum_{n=0}^{N_\varepsilon-1} |a_n|\,|z|^n}{e^{2\pi(M+\varepsilon)|z|}}$ is bounded at $|z|\to\infty$

hence let $A'_\varepsilon = \sup_{|z|\in\mathbb{C}} \dfrac{\sum_{n=0}^{N_\varepsilon-1} |a_n|\,|z|^n}{e^{2\pi(M+\varepsilon)|z|}}$

let $A_\varepsilon = 1 + A'_\varepsilon$.

Then

$$|f(z)| \leq \sum_{n=0}^{\infty} |a_n| \cdot |z|^n$$

$$= \sum_{n=0}^{N_\varepsilon - 1} |a_n| \cdot |z|^n + \sum_{N_\varepsilon}^{\infty} |a_n| \cdot |z^n|$$

$$\leq A_\varepsilon' \cdot e^{2\pi (M+\varepsilon)|z|} + 1 \cdot e^{2\pi (M+\varepsilon)|z|}$$

$$= A_\varepsilon \, e^{2\pi (M+\varepsilon)|z|}.$$

#9 (a) : Let $F(z)$ be a hol'c fcn on the right half plane. continuous on the boundary:

Suppose $|F(iy)| \leq 1 \quad \forall y \in \mathbb{R}$

and

$$|F(z)| \leq A e^{B|z|^\gamma} \quad \text{for } 0<\gamma<1$$

Then, show $|F(z)| \leq 1$.

pf : Let $\gamma < P < 1$. then $\forall \varepsilon > 0$. Let

$$F_\varepsilon(z) := F(z) \cdot e^{-\varepsilon z^P}$$

then $F_\varepsilon(z)$ is still hol'c on the right half plane.

$$|F_\varepsilon(z)| = |F(z)| \cdot e^{-\varepsilon r^P \cos(P\theta)} \qquad z = \cdot e^{i\theta}$$

$$\leq A e^{Bs^{\gamma} - \varepsilon \cdot s^{p} \cos(p\theta)}$$

say $\inf_{\theta \in (\frac{\pi}{2}, -\frac{\pi}{2})} \cos(p\theta) = C > 0$, then

$$\leq A e^{-\varepsilon C \cdot s^{p} + Bs^{\gamma}}$$

As $s \to +\infty$, the above bound $\to 0$.

Hence $|F_\varepsilon(z)| \to 0$ as $|z| \to \infty$ in the right half plane.

Rest of the proof is the same as in the book.