Gamelin $\overset{P356}{\#1,\ 10,\ 14}$, Stein #3, #4 (a)

1. Evaluate the following:

$$\bullet \quad \prod_{n=1}^{\infty}\left(1+\frac{1}{n(n+2)}\right) = \prod_{n=1}^{\infty}\frac{n^2+2n+1}{n(n+2)} = \frac{2^2}{1\cdot 3}\cdot\frac{3^2}{2\cdot 4}\cdot\frac{4^2}{3\cdot 5}\cdots$$

$$= 2$$

(we need to first observe that, since $\sum \frac{1}{n(n+2)} < \infty$, hence this product is convergent, thus we can do all these manipulations)

$$\bullet \quad \prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{\infty}\frac{n^2-1}{n^2} = \prod_{n=2}^{\infty}\frac{(n-1)(n+1)}{n^2}$$

$$= \frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdots\cdots = \frac{1}{2}$$

left over

$$\bullet \quad \prod_{n=3}^{\infty}\frac{n^2-1}{n^2-4} = \prod_{n=3}^{\infty}\frac{(n-1)(n+1)}{(n-2)(n+2)} = \frac{2\cdot 4}{1\cdot 5}\cdot\frac{3\cdot 5}{2\cdot 6}\cdot\frac{4\cdot 6}{3\cdot 7}\cdots$$

$$= 4$$

#10 Show that

$$\prod_{n=0}^{\infty}\left(1+z^{2^n}\right) = \frac{1}{1-z} \qquad \text{for } |z|<1$$

First, we observe that for $|z|<1$,

$$\sum |z|^{2^n} < \sum |z|^n < \infty$$

Hence the product converges.

Next, we only need to show that

$$(1-z)(1+z)(1+z^2)(1+z^4)\cdots = 1$$

Indeed, $(1-z)(1+z) = 1-z^2$

$(1-z^2)(1+z^2) = 1-z^4$

$\cdots\cdots$

More precisely, we may define

$$F_N = \prod_{n=0}^{N} (1+z^{2^n})$$

then $(1-z)F_N = 1-z^{2^{N+1}}$, hence

$$\left| F_N(z) - \frac{1}{1-z} \right| = \left| \frac{1-z^{2^{N+1}}}{1-z} - \frac{1}{1-z} \right| = \frac{|z|^{2^{N+1}}}{|1-z|}$$

For fixed $|z|<1$, we have $\lim_{N\to\infty} \left| F_N(z) - \frac{1}{1-z} \right| = 0$.

#14: Show that for $t>0$, we have

$$\lim_{m\to\infty} \prod_{\substack{-m<k<tm \\ k\neq 0}} \left(1+\frac{z}{k}\right) \longrightarrow \frac{\sin(\pi z)}{\pi z}\cdot t^z$$

Pf: Let $P_L(z) = \lim_{m\to\infty} \prod_{\substack{-m<k<tm \\ k\neq 0}} \left(1+\frac{z}{k}\right)$

$P_R(z) = \sin(\pi z)/\pi z.$

First, we need to show that the limit in the definition for $P_L(z)$ exists. We may consider

$$F_m(z) := \prod_{m \le k < tm} \left(1 + \frac{z}{k}\right)$$

then
$$\log F_m(z) = \sum_{m \le k < tm} \log\left(1 + \frac{z}{k}\right)$$
$$= \sum_{m \le k < tm} \left(\frac{z}{k} + R_k(z)\right)$$

we claim that
$$\lim_{m \to \infty} \sum_{m \le k < tm} R_k(z) = 0$$

Indeed, $\left|R_k(z)\right| = \left|\log\left(1 + \frac{z}{k}\right) - \frac{z}{k}\right| \le C \cdot \left|\frac{z}{k}\right|^2$
for some $c > 0$ and for $\left|\frac{z}{k}\right| < \frac{1}{2}$.

Thus $\displaystyle\sum_{m \le k < tm} \frac{|z|^2}{k^2} \to 0$ as $m \to \infty$, by Cauchy condition

on the convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Then
$$\sum_{m \le k < tm} \frac{1}{k} = \sum_{m \le k < tm} \frac{1}{k/m} \cdot \frac{1}{m} \xrightarrow{m \to \infty} \int_{x=1}^{t} \frac{1}{x}\, dx = \log t$$

This is because the limit of the sum is the Riemann integral.

Thus, we have
$$\lim_{m \to \infty} F_m(z) = e^{\lim_{m \to \infty} \log F_m(z)} = e^{z \cdot \log t} = t^z.$$

Stein #3 Show that, if τ is fixed with $\operatorname{Im}(\tau) > 0$, then the Jacobi Θ-function

$$\Theta(z|\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} \cdot e^{2\pi i \cdot nz}$$

has order of growth $= 2$

Pf: We note that

$$\Theta(z+1) = \Theta(z)$$

$$\Theta(z+\tau) = \sum_n e^{\pi i \tau [n^2 + 2n]} \cdot e^{2\pi i n z}$$

$$= e^{-\pi i \tau} \sum_n e^{\pi i \tau \cdot (n+1)^2 + 2\pi i n z}$$

$$= e^{-\pi i \tau - 2\pi i z} \Theta(z)$$

Hence $$|\Theta(z+\tau)| = |e^{-\pi i \tau - 2\pi i z}| \cdot |\Theta(z)|$$

$$= e^{\pi \cdot \operatorname{Im}(\tau) + 2\pi \cdot \operatorname{Im}(z)} \cdot |\Theta(z)|$$

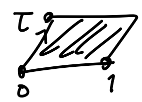
In fact,

$$\Theta(z+k\tau) = \sum_n e^{\pi i \tau \cdot n^2 + 2\pi i n (z + k\tau)}$$

$$= \sum_n e^{\pi i \tau (n^2 + 2nk + k^2 - k^2) + 2\pi i (n) z}$$

$$= e^{-\pi i \tau \cdot k^2 - 2\pi i k z} \cdot \Theta(z).$$

Let $F = [0,1] + [0,1]\cdot\tau \subset \mathbb{C}$



Let $M = \sup_{z\in F} |\theta(z)|$

Then for any $z \in F$, $\forall\, \alpha, \beta \in \mathbb{Z}$, we have

$$\sup_{z\in F} |\theta(z+\alpha+\beta\tau)| = \sup_{z\in F} |\theta(z+\beta\tau)| = \sup_{z\in F} \left[|\theta(z)| \cdot e^{\pi\beta^2 \mathrm{Im}(\tau)} \cdot e^{2\pi\cdot\beta\cdot \mathrm{Im}(z)} \right]$$

$$\leq M \cdot e^{\pi\beta^2 \mathrm{Im}(\tau) + 2\pi\beta\, \mathrm{Im}(\tau)}$$

$$\leq A\, e^{B\cdot\beta^2} \qquad \text{for some } A, B > 0.$$

We note that $\forall\, a, b \in \mathbb{R}$

$$|a+b\tau|^2 = (a + b\cdot \mathrm{Re}\,\tau)^2 + (b\cdot \mathrm{Im}\,\tau)^2$$

$$\geq b^2\cdot (\mathrm{Im}\,\tau)^2$$

Hence, for $z = a + b\tau$, $a \in [0,1]$, $b \in [0,1]$,

$$|z+\alpha+\beta\tau|^2 \geq (b+\beta)^2 \cdot (\mathrm{Im}\ \tau)^2$$

$$|\beta|^2 \leq \frac{1}{\mathrm{Im}(\tau)^2} |z+\alpha+\beta\tau|^2 + C.$$

Thus $\sup_{z \in F + \alpha + \beta\tau} |\theta(z)| \leq A e^{B|\beta|^2} \leq A e^{B\left(\frac{1}{\mathrm{Im}(\tau)^2}|z|^2 + C\right)}$

$\leq A' e^{B'|z|^2}$

since A', B' are independent of α, β. we have

$$|\theta(z)| \leq A' \cdot e^{B'|z|^2} \qquad \forall z.$$

Stein #4 (a)

Let $F(z) = \prod_{n=1}^{\infty} \left(1 - e^{-2\pi n t} \cdot e^{2\pi i z}\right)$, $t>0$

Show that $|F(z)| \leq A \cdot e^{B|z|^2}$ for some $A, B > 0$.

Pf :

[Hint: To prove (a), write $F(z) = F_1(z)F_2(z)$ where

$$F_1(z) = \prod_{n=1}^{N}(1 - e^{-2\pi n t}e^{2\pi i z}) \quad \text{and} \quad F_2(z) = \prod_{n=N+1}^{\infty}(1 - e^{-2\pi n t}e^{2\pi i z}).$$

Choose $N \approx c|z|$ with c appropriately large. Then, since

$$\left(\sum_{N+1}^{\infty} e^{-2\pi n t}\right) e^{2\pi|z|} \leq 1\,,$$

one has $|F_2(z)| \leq A$. However,

$$|1 - e^{-2\pi n t}e^{2\pi i z}| \leq 1 + e^{2\pi|z|} \leq 2e^{2\pi|z|}.$$

Thus $|F_1(z)| \leq 2^N e^{2\pi N|z|} \leq e^{c'|z|^2}$. Note that a simple variant of the function F arises as a factor in the triple product formula for the Jacobi theta function Θ, taken up in Chapter 10.]

We fill in the detail of the hint.

Let $c = \frac{1}{t}$, and fix some integer $N > c|z|$. then.

$$|F_2(z)| \leq \prod_{n=N+1}^{\infty} \left(1 + e^{-2\pi n t} \cdot e^{2\pi|z|}\right)$$

$$\leq e^{\sum_{n=N+1}^{\infty} e^{-2\pi n t + 2\pi|z|}}$$

$$= e^{e^{-2\pi(N+1)t + 2\pi|z|} \cdot \sum_{n=0}^{\infty} e^{-2\pi n t}}$$

$$\leq e^{\sum_{n=0}^{\infty} e^{-2\pi n t}} = e^{\frac{1}{1-e^{-2\pi t}}}$$

$$|F_1(z)| \leq \prod_{n=1}^{N} \left|1 - e^{-2\pi n t} \cdot e^{2\pi i z}\right|$$

$$\leq \prod_{n=1}^{N} \left(1 + |e^{-2\pi n t} \cdot e^{2\pi i z}|\right)$$

$$\leq \prod_{n=1}^{N} \left(1 + e^{2\pi|z|}\right)$$

$$\leq \prod_{n=1}^{N} \left(2e^{2\pi|z|}\right)$$

$$= 2^N e^{2\pi|z| \cdot N}$$

$$\leq e^{c_1|z|^2 + c_2|z|} \leq e^{c_3|z|^2}$$