Gamelin XIII.3 #15, #16 (a) (b), XII.4 #2,5

#15 Show that $\frac{1}{z} \prod_{n=1}^{\infty} \frac{n}{z+n} \left(\frac{n+1}{n}\right)^z$ converges to a meromorphic function whose pole at $0, -1, -2, \cdots$

Show that
$$\Gamma(z) = \lim_{m\to\infty} \frac{(m-1)!}{z(z+1)\cdots(z+m-1)} \cdot m^z$$

Show that
$$\Gamma(z+1) = z\,\Gamma(z).$$

Pf: (1)
$$\prod_{n=1}^{\infty} \frac{n}{z+n}\left(\frac{n+1}{n}\right)^z = \prod_{n=1}^{\infty} \left(\frac{1}{1+\frac{z}{n}}\right) \cdot e^{z\cdot \log\left(\frac{n+1}{n}\right)}$$

$$= \prod_{n=1}^{\infty} e^{-\log\left(1+\frac{z}{n}\right) + z\log\left(1+\frac{1}{n}\right)}$$

claim : ① $\underbrace{\sum_{n=1}^{\infty}}_{\text{if } z \neq -\mathbb{Z}} \left|\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right| < \infty$

② $\sum_{n=1}^{\infty} \left| -\frac{z}{n} + z\log\left(1+\frac{1}{n}\right)\right| < \infty$

① choose n_0 large enough, such that $\left|\frac{z}{n_0}\right| < \frac{1}{2}$.
Then for any $n \geq n_0$, we have

$$\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| = \left|\sum_{k=2}^{\infty} \left(\frac{z}{n}\right)^k / k\right|$$
$$\leq \left|\frac{z}{n}\right|^2 \cdot \sum_{k=2}^{\infty} \left|\frac{1}{2}\right|^{k-2} \cdot \frac{1}{k} \leq \left|\frac{z}{n}\right|^2 \cdot \sum_{j=0}^{\infty} \left|\frac{1}{2}\right|^j \leq 2\cdot\left|\frac{z}{n}\right|^2$$

Thus $\sum_{n=1}^{\infty} \left| \frac{z}{n} - \log\left(1+\frac{z}{n}\right) \right|$

$$= \sum_{n=1}^{n_0-1} \left| \frac{z}{n} - \log\left(1+\frac{z}{n}\right) \right| + \sum_{n=n_0}^{\infty} \left| \frac{z}{n} - \log\left(1+\frac{z}{n}\right) \right|$$

$$\leq \underbrace{\sum_{n=1}^{n_0-1} \left| \frac{z}{n} - \log\left(1+\frac{z}{n}\right) \right|}_{\text{finitely many terms.}} + \underbrace{\sum_{n=n_0}^{\infty} 2 \cdot \left|\frac{z}{n}\right|^2}_{< 2|z^2| \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty}$$

$$< \infty$$

② $\sum_{n=1}^{\infty} \left| -\frac{z}{n} + z \log\left(1+\frac{1}{n}\right) \right| = |z| \cdot \underbrace{\sum_{n=1}^{\infty} \left| -\frac{1}{n} + \log\left(1+\frac{1}{n}\right) \right|}_{\text{apply ① with } z \text{ replaced by } 1.} < \infty$

This finishes the claim, thus.

$$\prod_{n=1}^{\infty} e^{-\log\left(1+\frac{z}{n}\right) + z \log\left(1+\frac{1}{n}\right)} = \prod_{n=1}^{\infty} e^{-\log\left(1+\frac{z}{n}\right) + \frac{z}{n}} \cdot e^{-\frac{z}{n} + z\log\left(1+\frac{1}{n}\right)}$$

$$= e^{\sum_n -\log\left(1+\frac{z}{n}\right) + \frac{z}{n} + \sum_n -\frac{z}{n} + z \log\left(1+\frac{1}{n}\right)}$$

since both sums are absolutely convergents, we have the product converges.

The product expansion for Γ follows from the partial product:

$$\frac{1}{z}\prod_{n=1}^{m-1}\left(\frac{n}{n+z}\right)\cdot\left(\frac{n+1}{n}\right)^{z}$$

$$= \frac{1}{z}\frac{(m-1)\cdots 1}{(1+z)\cdots(z+m-1)}\, m^{z}$$

- $$\Gamma(z+1) = \lim_{m\to\infty}\frac{(m-1)\cdots\cdots 1}{(z+1)\cdots(z+m)}\, m^{z+1}$$

$$= \lim_{m\to\infty} z\cdot\ \frac{1}{z}\frac{m(m-1)\cdots\ 1}{(z+1)\cdots(z+m)}\cdot (m+1)^{z}\left(\frac{m}{m+1}\right)^{z}$$

$$= z\cdot\Gamma(z)\cdot\lim_{m\to\infty}\left(\frac{m}{m+1}\right)^{z}$$

$$= z\cdot\Gamma(z).$$

#16 Let α_k be a seq of complex numbers, maybe with repetition. $|a_k|<1$, $|a_k|\to 1$. Consider

$$B(z) = \prod_{k=1}^{\infty}\frac{\overline{\alpha_k}}{|\alpha|}\cdot\frac{\alpha_k - z}{1-\overline{\alpha_k}\cdot z}$$

(a) If $\sum(1-|a_k|)<\infty$, Let E be the set of accumulation pts on $\partial\mathbb{D}$. Show that this infinite product converges on $\mathbb{C}^*\setminus E$. to a meromorphic function $B(z)$. s.t.

$$|B(z)|<1 \qquad \text{for } z\in\mathbb{D}$$

$|B(z)| = 1$ for $z \in \partial\mathbb{D} \setminus E$.

and $B(z)$ has zero exactly at points a_k.

(b) Show that, if $\sum (1-|a_k|) = +\infty$, then

the infinite product converges uniformly on compact subsets of $\mathbb{D}$ to 0.

Pf: (a) suppose the integer N is large enough, such that $\forall\ n > N$, $(1 - |a_n|) < \frac{1}{2}$

For any $\alpha \in \mathbb{C}$, with $\frac{1}{2} < |\alpha| < 1$, any $|z| < 1$, we have

let $\alpha = r \cdot e^{i\theta}$

$$\frac{\bar{\alpha}}{|\alpha|} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z} = e^{-i\theta} \frac{re^{i\theta} - z}{1 - re^{-i\theta} z} = r \cdot \frac{1 - r^{-1} \cdot e^{-i\theta} z}{1 - r\ e^{-i\theta} z}$$

$$= r \cdot \left(\frac{1 - re^{-i\theta} z + (r - r^{-1}) e^{-i\theta} z}{1 - re^{-i\theta} z} \right)$$

$$= r \cdot \left(1 + (r - r^{-1}) \cdot \frac{e^{-i\theta} z}{1 - re^{-i\theta} z} \right)$$

Now, $\prod_{n=1}^{\infty} |a_n| = \prod_{n=1}^{\infty} (1 - (1 - |a_n|))$ $\because \sum (1 - |a_n|) < \infty$

$< \infty$

If $|z|<1$, then $\left|1-re^{-i\theta}z\right| \geq 1-r|z| \geq 1-|z|$.

Hence $\prod_{n=1}^{\infty} \left(1+(r_n-r_n^{-1}) \cdot \frac{e^{-i\theta_n} \cdot z}{1-r_n e^{-i\theta_n} z}\right)$

$$\sum_{n=1}^{\infty} \left|\left(r_n-r_n^{-1}\right) \frac{e^{-i\theta_n} z}{1-r_n \cdot e^{-i\theta_n} z}\right| \leq \frac{|z|}{1-|z|} \cdot \sum_{n=1}^{\infty} |r_n^{-1}-r_n|$$

$$= \frac{|z|}{1-|z|} \sum_{n=1}^{\infty} \left(|r_n^{-1}-1|+|1-r_n|\right)$$

$\because \sum_{n=1}^{\infty} (1-r_n) < \infty$.

and since $r_n \to 1$, $\therefore \exists N$ s.t. $\forall n > N$, $r_n > \frac{1}{2}$.

$$\sum_{n=1}^{\infty} |r_n^{-1}-1| = \sum_{n=1}^{\infty} \left|\frac{1-r_n}{r_n}\right|$$

$$\leq \sum_{n=1}^{N} \frac{1-r_n}{r_n} + \sum_{n=N+1}^{\infty} \frac{1-r_n}{(1/2)} < \infty$$

Hence, the above $\sum_{n=1}^{\infty} \left|\left(r_n-r_n^{-1}\right) \frac{e^{-i\theta_n} z}{1-r_n \cdot e^{-i\theta_n} z}\right| < \infty$

$$\therefore \prod_{n=1} \left|\left(1+(r_n-r_n^{-1}) \cdot \frac{e^{-i\theta_n} \cdot z}{1-r_n e^{-i\theta_n} z}\right)\right| < \infty$$

If $|z|>1$, let $w=\frac{1}{z}$, let $\beta_n = \overline{\alpha_n}$, then

$$B(z) = \prod_n \frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - \frac{1}{w}}{1-\overline{\alpha_n}\frac{1}{w}} = \prod_n \frac{\overline{\alpha_n}}{|\alpha_n|} \frac{w\alpha_n - 1}{w-\overline{\alpha_n}}$$

$$= \left[\prod_n \frac{\overline{\beta_n}}{\beta_n} \frac{\beta_n - w}{1 - \overline{\beta_n} w} \right]^{-1}$$

Hence, by the same argument, this product converges for $|w| < 1$.

Finally, if $|z| = 1$, and $z \notin E$, then $\exists \varepsilon > 0$, s.t. $B_\varepsilon(z) \cap E = \phi$. And $\exists N > 0$, s.t. $\forall n > N$, $|\alpha_n - z| > \frac{\varepsilon}{2}$.

Thus.
$$\left| \frac{1}{1 - \overline{\alpha}_n z} \right| = \left| \frac{1}{z - \alpha_n} \right| < \frac{2}{\varepsilon}$$

thus

$$\sum_{n=N}^{\infty} \left| \left(r_n - r_n^{-1} \right) \frac{e^{-i\theta_n} z}{1 - r_n \cdot e^{-i\theta_n} z} \right|$$

$$\leq |z| \cdot \frac{2}{\varepsilon} \cdot \sum_{n=N}^{\infty} \left| r_n - r_n^{-1} \right| < \infty$$

Hence $B(z)$ also converges on $z \in \partial \mathbb{D} \setminus E$. In particular since $\left| \frac{\alpha - z}{1 - \overline{\alpha} z} \right| = 1$, each Blasche factor has unit modulus, thus $|B(z)| = 1 \quad \forall z \in \partial \mathbb{D} \setminus E$.

(b) If $\sum_n (1 - |a_n|) = +\infty$, then

$$\prod_n |\alpha_n| = 0$$

Since $\left| \dfrac{\alpha - z}{1-\bar{\alpha} z} \right| \leq |\alpha|^{\frac{1}{2}}$, $\forall |\alpha|<1, |z|<1$.

hence $\prod_n \left| \dfrac{\alpha_n - z}{1-\bar{\alpha}_n z} \right| \leq \prod_n |\alpha_n|^{\frac{1}{2}} = 0.$

[G] P360 #2.

$$f(z) = z\prod_{n=1}^{\infty}\left(1-\frac{z}{n^2}\right)$$

convergent $\because \sum \frac{1}{n^2} < \infty$

#5: $$\prod_{\alpha \in \mathbb{Z}+i\mathbb{Z}} \left(1-\frac{z}{\alpha}\right) e^{\left(\frac{z}{\alpha}\right)+\left(\frac{z}{\alpha}\right)^2/2}$$

or if you want to be safe, use. $\prod E_n(\alpha_n)$

where α_n is an enumeration of $\mathbb{Z}+i\mathbb{Z}$.