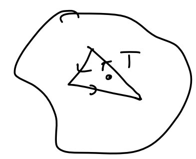
HW 4 #6, 7, 8, 10, 12

#6. Let f be a holomorphic function in $\Omega \setminus \{w\}$, and is bounded near w. Let T be a triangle in Ω containing w, then

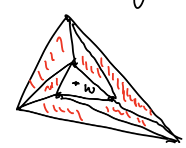
$$\int_T f(z)\, dz = 0$$



Pf: without loss of generality, we may assume $w = 0$.

For $1 > \varepsilon > 0$, let εT denote the triangle T rescaled by ε.

Then, we may triangulate the polygon region between T and εT as shown.



By Goursat theorem, integral of $f(z)$ along any of the shaded triangles is zero. Hence.

$$\int_T f(z)\, dz = \int_{\varepsilon T} f(z) \cdot dz. \qquad \forall\ 1 > \varepsilon > 0.$$

However, since f is bounded on the solid triangle bounded by T,

$$\left| \int_T f(z)\, dz \right| = \left| \int_{\varepsilon T} f(z)\, dz \right| \leq C \cdot \varepsilon \cdot \text{length}(T) \qquad \forall\ \varepsilon > 0$$

Thus. $\int_T f(z)\, dz = 0$.

Problem (7). *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image satisfies*

$$2|f'(0)| \leq d$$

Moreover it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Solution. For $0 < r < 1$, let C_r be the circle centered at 0 with radius r. Consider the Cauchy integral expression for $f'(0)$, we have

$$f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{(w-0)^2}dw.$$

We may replace the integration variable w by $-w$, and get

$$f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(-w)}{(-w-0)^2}d(-w).$$

Summing up the two equations, we have

$$2f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)-f(-w)}{w^2}dw.$$

Taking absolute value on both sides, we have

$$\begin{aligned}
2|f'(0)| &\leq \frac{1}{2\pi}\int_{C_r} \frac{|f(w)-f(-w)|}{|w|^2}|dw| \\
&\leq \frac{1}{2\pi}\sup_{w\in C_r} |f(w)-f(-w)| \cdot \int_{C_r} \frac{1}{|w|^2}|dw| \\
&\leq \frac{1}{2\pi} d \frac{2\pi r}{r^2} \\
&= \frac{d}{r}
\end{aligned}$$

Since the inequality holds for any $0 < r < 1$, we get

$$2|f'(0)| \leq \inf_{0<r<1} \frac{d}{r} = d.$$

Problem (8). *If f is holomorphic on a strip $\{x+iy \mid x \in \mathbb{R}, -1 < y < 1\}$, with*

$$|f(z)| \le A(1+|z|)^\eta, \qquad \eta \text{ a fixed real number}$$

for all z in that strip. Show that for each $n \ge 0$, there exists a constant A_n, such that

$$|f^{(n)}(x)| \le A_n(1+|x|)^\eta$$

for all $x \in \mathbb{R}$.

Solution. Fix a r with $0 < r < 1$. Let $C_r(x)$ be the circle centered at x with radius r. Then by Cauchy estimate

$$|f^{(n)}(x)| \le \frac{n!}{r^n} \sup_{w \in C_r(x)} |f(w)| \le \frac{n!}{r^n} A \sup_{w \in C_r(x)} (1+|w|)^\eta.$$

We claim that there exists a constant C, only dependent on η, such that

$$\sup_{w \in C_r(x)} (1+|w|)^\eta < C(1+|x|)^\eta$$

Given the claim, we have the desired result

$$|f^{(n)}(x)| \le \inf_{0<r<1} \frac{n!}{r^n} AC(1+|x|)^\eta = n!AC(1+|x|)^\eta,$$

with $A_n = n!AC$.

Now we prove the claim. In fact we show one can take $C = 2^{|\eta|}$. Indeed, if $\eta > 0$, then

$$\sup_{w \in C_r(x)} (1+|w|)^\eta \le (1+|x|+r)^\eta \le (2+|x|)^\eta = 2^\eta(1+|x|/2)^\eta \le 2^\eta(1+|x|)^\eta.$$

If $\eta < 0$, then

$$\sup_{w \in C_r(x)} (1+|w|)^\eta \le \begin{cases} 1 & |x| < r \\ (1+|x|-r)^\eta & |x| \ge r \end{cases} \le \begin{cases} 1 & |x| < 1 \\ |x|^\eta & |x| \ge 1 \end{cases}$$

Let $h(x)$ be the piecewise defined function on the right in the above inequality. For $|x| < 1$,

$$\sup_{|x|<1} \frac{h(x)}{(1+|x|)^\eta} = \sup_{|x|<1} \frac{1}{(1+|x|)^\eta} = 2^{-\eta}$$

#10 Weierstrass Thm says every continuous function on [0,1] can be uniformly approx. by polynomial functions. Can continuous function on $\overline{\mathbb{D}}$ be uniformly approx by polynomial functions?

Ans: No. If $f: \overline{\mathbb{D}} \to \mathbb{C}$ is not holomorphic, then f cannot be uniformly approximated by a sequence of holomorphic functions, since uniform limit of hol'c functions is holomorphic.

See Thm 1.5.2 in Stein

5.2 Sequences of holomorphic functions

Theorem 5.2 *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.*

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem implies

$$\int_T f_n(z)\,dz = 0 \quad \text{for all } n.$$

By assumption $f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\int_T f_n(z)\,dz \to \int_T f(z)\,dz.$$

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As a result, we find $\int_T f(z)\,dz = 0$, and by Morera's theorem, we conclude that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω.

Problem (12). *Let u be a real valued function defined on the unit disk $\mathbb{D}$. Suppose that u is twice differentiable and harmonic, that is $\Delta u(x,y) = 0$ for all $x, y \in \mathbb{D}$.*

(a) Prove that there exists a holomorphic function f on the unit disk, such that

$$Re(f) = u$$

(b) Deduce from this result, the Poisson integration formula. If u is harmonic in $\mathbb{D}$ is is continuous on its closure $\overline{\mathbb{D}}$, then if $z = re^{i\theta}$, one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(e^{i\varphi})d\varphi$$

where $P_r(\theta)$ is the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

Solution. (a) Let's make some observation first. To construct f, we try to construct its derivative f' then integrate to get f. If we know a holomorphic function $f(z) = u(z) + iv(z)$, then $f'(z) = 2\partial_z u(z)$, indeed

$$\partial_z f(z) = \partial_x f(x,y) = \partial_x(u(x,y) + iv(x,y)) = \partial_x u - i\partial_y u = 2\partial_z u(z).$$

Now we begin the proof. Define $g(z) = 2\partial_z u(z) = \partial_x u - i\partial_y u$. Then $g(z)$ is once differentiable (though $\partial_x g, \partial_y g$ may not be continuous), since

$$\partial_{\bar{z}} g(z) = 2\partial_{\bar{z}}\partial_z u(z) = (1/2)\Delta u = 0$$

hence $g(z)$ is holomorphic for all $z \in \mathbb{D}$. From Theorem 2.1, we know g has a primitive F. We claim that $\mathrm{Re}F - u$ is a constant. Indeed, we have

$$\partial_x(\mathrm{Re}F - u) + i\partial_y(\mathrm{Re}F - u) = 2\partial_z(\mathrm{Re}F - u) = g(z) - g(z) = 0,$$

hence the partial derivatives of $(\mathrm{Re}F - u)$ vanishes, hence $\mathrm{Re}F - u$ is a constant. Denote this constant by c, and define $f = F - c$, we then get f a holomorphic function with $\mathrm{Re}f = u$.

(b) Apply (a) to get a holomorphic function f with $\mathrm{Re}f = u$. If $z = e^{i\theta}$, then let $R \in \mathbb{R}$ such that $|z| < R < 1$. Then by Exercise 11, we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right)d\varphi$$

Taking the real part on both sides, we get

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\frac{R^2 - r^2}{R^2 - 2Rr\cos(\varphi - \theta) + r^2}d\varphi.$$

Let $R \to 1$, by uniform continuity, we get

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})P_r(\theta - \varphi)d\varphi.$$

Note that $\cos(x)$ is an even function, hence $P_r(\theta - \varphi) = P_r(\varphi - \theta)$.