Things that we have covered since midterm 1

① Rouché theorem, open mapping, maximum principle

② Fourier transformation. Relationship between analyticity of $f(z)$, and properties of $\hat{f}(\xi)$

- $f(z)$ is defined for $\{|\mathrm{Im}(z)| < a\}$
  $\Leftrightarrow$ $\hat{f}(\xi)$ decays like $e^{-2\pi a|\xi|}$

(Paley-Wiener)
- $f(z)$ is entire and has growth bounded by $|f(z)| < e^{2\pi M|z|}$
  $\hat{f}(\xi)$ vanishes for $\xi \in \mathbb{R}$ and $|\xi| > M$.

③ Entire function and infinite product.

- Order of growth and distribution of zero.

- Jensen's formula

$$\sum \log\left|\frac{a_n}{R}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)|$$

a_n roots of $f(z)$ in $D_R(0)$

- If order of growth $\leq \rho$, then
  $\#\{\text{roots of } f(z) \text{ in } D_R(0)\} \leq C \cdot R^{\rho}$

- <u>infinite product</u>:
  - Weierstrass Formula: construct function with prescribed roots.
  - Hadamard Factorization: for functions of finite order growth.

(did not prove in class)

④ Γ- function

- integral presentation, analytic continuation.
- relation with sine function.

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Sample Midterm 2 Questions:

(1) if $f_0(z)$ is a polynomial with roots $z=1, z=2$ and $f_1(z)$ ———— with roots $z=3, z=4$. and if we define $f_t(z) = (1-t)\cdot f_0(z) + t\, f_1(z)$, then $f_t(z)$ will have two roots, moving from $\{1,2\}$ continuously to $\{3,4\}$, true or false?

(2). If $f(z) = \dfrac{(z-a_1)\cdots(z-a_n)}{(z-b_1)\cdots(z-b_n)}$, with a_i, b_j all distinct ($a_i \neq b_j$ $\forall i,j$), then f defines a holomorphic function: $\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ (i.e. a rational function), what is $f(\infty) = ?$
what is $f^{-1}(\infty) = ?$

(3) Let $f(z) = z^2$, $\Omega = \{\operatorname{Im} z > 0\}$.
is $f(\Omega)$ an open set? What is $f(\Omega)$?

(4). Let $f(z) = \dfrac{1}{z^2+a^2}$, what is $\hat{f}(\xi)$?

(5) Let $f(z) = e^{-z^4}$, what can we say

about $\hat{f}(\xi)$? Does it exist ?

Is $\hat{f}(\xi)$ a hol'c function in ξ ?

(6) For $|q|<1$, consider the infinite product

$$\varphi_q(z) = \prod_{n=1}^{\infty} (1 - q^n z)$$

- is the product convergent ?
- what's the zero and poles for $\varphi_q(z)$?
- what's the order of growth for $\varphi_q(z)$?