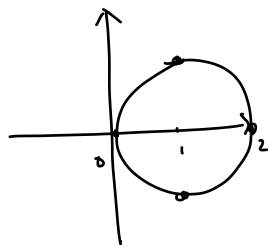
1. Let $f(z) = z^2$. Sketch.

(10 pts)

(1) $C = \{ |z-1| = 1 \}$

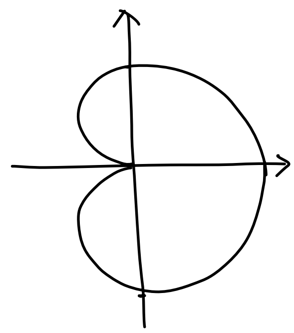
(2) $f(C)$

(1)



C

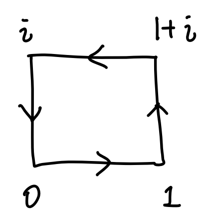
(2)



f(C)

2. Let γ be the contour :

(20 pts) Compute the following integrals.



(a) $\int_\gamma |dz| = 4$

(b) $\int_\gamma \frac{1}{z - \frac{1}{2}(1+i)} \, dz = 2\pi i$.

(c) $\int_\gamma z \cdot dz = 0$

(d) $\int_\gamma x \cdot dz = \int_\gamma x \, dx + \int_\gamma x \, d(iy) = 0 + i\int_\gamma x \, dy$

$$= i \int_{\boxed{\ }} dx\,dy = i \cdot 1 = i$$

3 (20 pts) Compute the following contour integrals

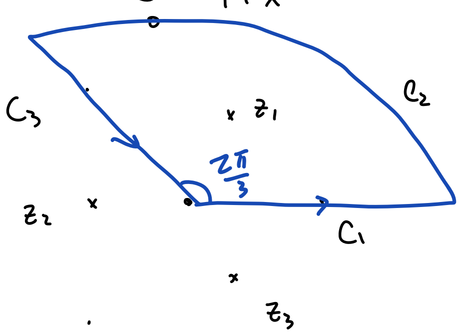
- $$\frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})}\,dz = 0 \qquad (10 \text{ pts})$$

method 1 : Residue thm : integral $= \operatorname{res}_{z=\frac{1}{2}} \left(\frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})} \right)$

$$+ \operatorname{res}_{z=\frac{1}{3}} \left(\frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})} \right)$$

$$= \frac{1}{\frac{1}{2}-\frac{1}{3}} + \frac{1}{\frac{1}{3}-\frac{1}{2}} = 0$$

- $$I = \int^{\infty} \frac{1}{1+x^3}\,dx \qquad (10 \text{ pts})$$



$z^3+1=0$ has roots at

$$z = e^{\frac{\pi i}{3}},\ e^{\frac{\pi i}{3}+\frac{2\pi i}{3}},\ e^{\frac{\pi i}{3}+\frac{4\pi i}{3}}$$
$$\quad\ \ z_1 \qquad\qquad z_2 \qquad\qquad z_3$$

The above contour $I(R) \int_{C_1+C_2+C_3} \frac{1}{1+z^3}\,dz = 2\pi i \cdot \operatorname{Res}_{z=z_1} \frac{1}{1+z^3}$

$$= 2\pi i \left.\frac{1}{(1+z^3)'}\right|_{z=z_1} = 2\pi i \cdot \frac{1}{3\cdot(e^{\frac{\pi i}{3}})^2}$$

$$= 2\pi i \frac{1}{3e^{2\pi i/3}}$$

$$I_1(R) = \int_0^r \frac{1}{1+r^3}\,dr$$

$$|I_2(R)| \le \int_{C_2} \left|\frac{1}{1+z^3}\right| |dz| \le C \cdot \frac{1}{R^2} \quad \text{for some } C$$

$$I_3(R) = \int_{C_3} \frac{1}{1+z^3}\,dz = \int_{r=R}^{r=0} \frac{1}{1+(e^{i2\pi/3}\cdot r)^3}\, d(e^{i2\pi/3}\cdot r)$$

$$= \int_{r=R}^{r=0} \frac{1}{1+r^3}\, e^{i\frac{2\pi}{3}}\, dr$$

$$= -e^{i2\pi/3}\cdot I_1(R)$$

Hence $I(R) = I_1(R) + I_2(R) + I_3(R)$,

$$\lim_{R\to\infty} I(R) = 2\pi i \frac{1}{3e^{2\pi i/3}}$$

$$\lim_{R\to\infty} I_2(R) = 0, \quad \lim_{R\to\infty} I_1(R) = I, \quad \lim_{R\to\infty} I_3(R) = -e^{\frac{i2\pi}{3}} I$$

thus $$I - e^{\frac{2\pi i}{3}} I = 2\pi i \frac{1}{3e^{2\pi i/3}} = \frac{2\pi}{3} e^{-\frac{1}{6}\pi i}$$

$$1 - e^{\frac{2\pi i}{3}} = e^{\frac{\pi i}{3}}\left(e^{-\frac{\pi i}{3}} - e^{\frac{\pi i}{3}}\right) = e^{\frac{\pi i}{3}}(-2i)\sin\left(\frac{\pi}{3}\right)$$

$$= 2\sin\left(\frac{\pi}{3}\right)\cdot e^{-\frac{\pi}{6}i}$$

$$\therefore \quad I = \frac{\frac{2\pi}{3}\cdot e^{-\frac{1}{6}\pi i}}{2\sin\left(\frac{\pi}{3}\right) e^{-\frac{\pi}{6}i}} = \frac{\pi/3}{\sin(\pi/3)}$$