① Review about topology (cont'd)
② Definition of hol'c function.

Last time :

- open set : Ω is an open set if $\forall z \in \Omega$ $\exists D_r(z) \subset \Omega$
- closed set : A is a closed set, if A contains all of its limit point.

Fact : Ω is open $\iff$ Ω^c is closed.

- bounded set : Ω is bounded if

$$\mathrm{diam}(\Omega) := \sup_{z,w \in \Omega} \mathrm{dist}(z,w) < \infty$$

$$\mathrm{dist}(z,w) = |z-w|$$

- compact subset of $\mathbb{C}$:
  A subset $A \subset \mathbb{C}$ is compact if A is closed and bounded.

More generally : if X is a metric space, we say

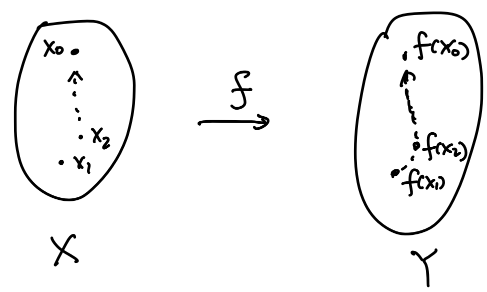
- X is sequentially compact, if every seq in X has convergent subseq.
- X is compact, if every open cover of X, (ie. $X = \bigcup_{\alpha \in I} U_\alpha$) admits a finite subcover, (i.e. there is a finite subcollection $\tilde{I} \subset I$, $X = \bigcup_{\alpha \in \tilde{I}} U_\alpha$)

- <u>connected subset</u>: X is connected, if X cannot be written as a union of 2 disjoint open subsets.

- <u>Continuous Maps</u>: X, Y are metric spaces.
  - Let $f: X \to Y$ (e.g. $X = \mathbb{R}$, $Y = \mathbb{R}$).

we say f is <u>continuous at point $x_0 \in X$</u>, if

$$f(x_0) = \lim_{x_n \to x_0} \underbrace{f(x_n)}_{\text{in } Y}.$$



- If f is continuous at every point of X, we say f is a continuous map $X \to Y$.

- Equivalently, we say f is cont. at $x_0 \in X$, if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall x \in B_\delta(x_0)$, we have $f(x) \in B_\varepsilon(f(x_0))$.

- ☆ Equivalently; if for any open subset $V \subset Y$, $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is an open subset of X. then we say, $f: X \to Y$ is continuous.

  'preimage of open is open'

Prop: If $f: X \to Y$ is continuous, and $A \subset X$ is compact, then $f(A) \subset Y$ is compact.

Holomorphic Function:

- Let $\Omega \subset \mathbb{C}$ be any subset.
  $f: \Omega \to \mathbb{C}$ any function.

  We say f is holomorphic at $z_0 \in \Omega$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{is well-defined.}$$

(i.e. limit exists.)

To be more precise: we define

$$u: \Omega \setminus \{z_0\} \to \mathbb{C}.$$

$$u(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

we can ask, if $\lim_{z \to z_0} u(z)$ exist.

If the limit is well defined, we call that limit the (complex) derivative, $f'(z_0)$.

- If $\Omega \subset \mathbb{C}$ is open, and f is hol'c at every pt in Ω, we say $f: \Omega \to \mathbb{C}$ is a hol'c function on Ω.

Ex 1:
- $f(z) = z$, it is hol'c on $\mathbb{C}$ (an entire function)

$\forall z_0 \in \mathbb{C}$, $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = \lim \dfrac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1.$

$\Rightarrow f'(z_0) = 1$

$a^2 - b^2 = (a+b)(a-b)$

- $f(z) = z^2$,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0.$$

$f'(z_0) = 2z_0$

- $\cdots$ $f(z) = z^n$, $\Rightarrow f'(z_0) = n \cdot z_0^{n-1}$.

- One also have polynomials

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ $\quad a_i \in \mathbb{C}$ const.

$p(z)$ is also hol'c.

Ex2: $f(z) = \frac{1}{z}$. $\quad f: \underbrace{\mathbb{C} \setminus \{0\}}_{= \mathbb{C}^*} \to \mathbb{C}$

For any $z_0 \in \mathbb{C}^*$, we have.

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0} = \frac{\frac{z_0 - z}{z z_0}}{z - z_0}$$

$$= -\frac{1}{z z_0} \xrightarrow{\text{as } z \to z_0} -\frac{1}{z_0^2}.$$

$f'(z_0) = -\frac{1}{z_0^2}$

$\Rightarrow f$ is hol'c on $\mathbb{C} \setminus \{0\}$.

Ex: $f(z) = \bar{z}$ $\quad$ It is not hol'c at any point in $\mathbb{C}$.

$\forall z_0 \in \mathbb{C}$,

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{\bar{h}}{h}$$

let $h = z - z_0$

as h goes to 0 along different direction, say
$h = r\cdot e^{i\theta}$, for some fixed θ, and $r \to 0_+$.

we have $\bar{h} = r\cdot e^{-i\theta}$.

$$\frac{\bar{h}}{h} = \frac{r\cdot e^{-i\theta}}{r\cdot e^{i\theta}} = e^{-2i\theta}$$

the result depends on the direction that h goes to 0.
Thus, $\bar{z}$ is not hol'c.

<u>More non example</u>:

- $Re(z) = \frac{1}{2}(z+\bar{z})$ is <u>not hol'c.</u>
  $Im(z) = \frac{1}{2i}(z-\bar{z})$.

- $|z|^2 = z\cdot\bar{z}$ not hol'c.

- $|z+z^2|$ not hol'c.

<u>Cauchy-Riemann Equation</u>:

- Let $f(z): \Omega \to \mathbb{C}$ be written as a real vector valued fcn.
  $F(x,y): \Omega \to \mathbb{R}^2$
  $(x,y) \mapsto (u,v)$. $u(x,y)$, $v(x,y)$.

- We say F is <u>differentiable</u> at point $(x_0,y_0) \in \Omega$, if there exists a 2×2 matrix, J, such that

- $F(x_0+h_1,\ y_0+h_2) = F(x_0, y_0) + J\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + |h|\cdot R(h)$

$\begin{pmatrix} u \\ v \end{pmatrix}$ a colum vector.

where $R(h) \to 0$, as $|h| \to 0$.

$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, $|h| = \sqrt{h_1^2 + h_2^2}$.

- <u>Prop</u>: if f is hol'c at $z_0 = x_0 + i y_0 \in \Omega$, then F is differentiable at z_0, and the partial derivatives satisfies.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

<u>Pf</u>: f hol'c at z_0

$r = r_1 + i r_2$

$$\Leftrightarrow \quad f(z_0+h) = f(z_0) + f'(z_0)\cdot h + |h|\cdot r(h)$$

$h = h_1 + i h_2$ where $r(h) \to 0$. as $h \to 0$.

Take the real & imaginary part above, we get.

$$\begin{cases} u(z_0+h) = u(z_0) + \mathrm{Re}(f'(z_0))\cdot h_1 - \mathrm{Im}(f'(z_0))\cdot h_2 \\ \qquad\qquad + |h|\cdot r_1(h) \\ v(z_0+h) = v(z_0) + \mathrm{Re}(f'(z_0))\cdot h_2 + \mathrm{Im}(f'(z_0))\cdot h_1 \\ \qquad\qquad + |h|\, r_2(h). \end{cases}$$

$(a+ib)\cdot(c+id) = (ac-bd) + i(ad+bc)$.

$\Rightarrow$ for. $J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$, we have.

$$\begin{pmatrix} u(z_0+h) - u(z_0) \\ v(z_0+h) - v(z_0) \end{pmatrix} = J\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + |h|\cdot\begin{pmatrix} r_1(h) \\ r_2(h) \end{pmatrix}.$$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}.$$

if $h = h_1 + i h_2$, and $h_2 = 0$, then.

$$f'(z_0) = \lim_{\substack{h_1 \to 0 \\ h_1 \in \mathbb{R}}} \frac{f(z_0 + h_1) - f(z_0)}{h_1} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

if $h_1 = 0$, let $h_2 \to 0$. then.

$$f'(z_0) = \lim_{\substack{h_2 \to 0 \\ h_2 \in \mathbb{R}}} \frac{f(z_0 + ih_2) - f(z_0)}{ih_2} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right).$$

setting $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

and compare the real & imaginary of both sides, we get.

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y}. \end{cases}$$

and $f'(z_0) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$.

$$Re(f'(z_0)) = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$Im(f'(z_0)) = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

$\square$.

Conversely :

Prop : If F is continuously differentiable at every pts. in Ω, and Cauchy Riemann eq are satisfied, then $f(z)$ is a hol'c on Ω.

Pf: see Stein.

Next time: e^x, $\log(x)$, $\sqrt{z}$, $\sqrt{(z-1)(z-2)}$, ...

power series.