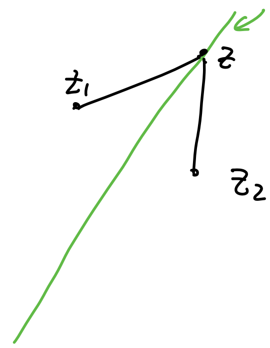
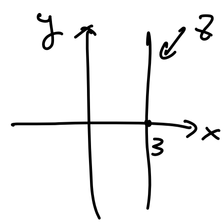
1. (1) $|z-z_1| = |z-z_2|$

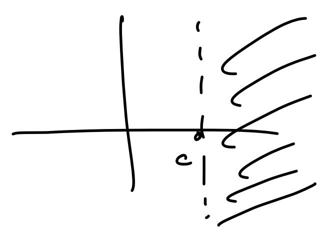


(2) $\frac{1}{z} = \bar{z}$, $\iff$ $z\bar{z} = 1$ $\iff$ $|z| = 1$

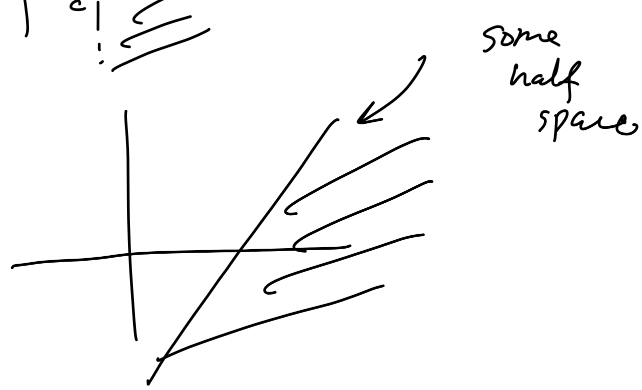
(3) $\mathrm{Re}(z) = 3$



(4) $\mathrm{Re}(z) > c$

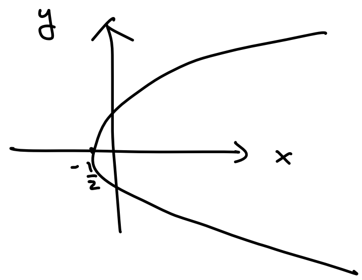


(5) $\mathrm{Re}\,(az+b) > 0$

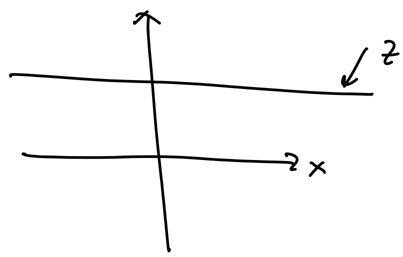


(6) $|z| = \mathrm{Re}(z) + 1$ $\quad x^2 + y^2 = (x+1)^2$

$y^2 = 2x+1$



(7) $\mathrm{Im}(z) = c$, $c \in \mathbb{R}$.



#2 show that

$$\langle z, w \rangle = \frac{1}{2}\left[(z,w) + (w,z) \right] = \mathrm{Re}\,(z,w)$$

Euclidean innerproduct

Pf: $\because$ $(z,w) = \overline{(w,z)}$

$\therefore$ $\mathrm{Re}\,(z,w) = \frac{1}{2}\left((z,w) + (w,z) \right)$

say $z = a+ib$, $w = c+id$, then.

$$(z,w) = z\overline{w} = (a+ib)(c-id)$$

$$= ac + bd + i(bc - ad)$$

$$\operatorname{Re}(z, w) = ac+bd = \langle z, w\rangle \quad \checkmark.$$

#3 If $s=0$, there is one sol'n. $w=0$

If $s>0$, there are n sol'n.

$$w = s^{\frac{1}{n}} \cdot e^{\frac{i\varphi}{n} + \left(\frac{2\pi i}{n}\right)\cdot k} \qquad k=0,1,\cdots,n-1$$

(well-defined for $s>0$)

#7 (a) Prove that $\forall\, z, w \in \mathbb{C}$, with $z\bar{w} \neq 1$.

$$|z|<1,\ |w|<1 \quad \Rightarrow \quad \left|\frac{w-z}{1-\bar{w}z}\right| < 1$$

and

$$|w|=1 \text{ or } |z|=1 \quad \Rightarrow \quad \left|\frac{w-z}{1-\bar{w}z}\right| = 1$$

Pf: (a) Following the hint, assume $z = r\cdot e^{i\theta}$, then

$$\left|\frac{w-z}{1-\bar{w}z}\right| = \left|\frac{w-re^{i\theta}}{1-\bar{w}\cdot r\cdot e^{i\theta}}\right| = \left|\frac{u-r}{1-\bar{u}\cdot r}\right|$$

where $u = w \cdot e^{-i\theta}$, (hence $\bar{u} = \bar{w} \cdot e^{i\theta}$).

so suffice to assume $|z| \geq 0$.

$$(u-r)(\bar{u}-r) = |u|^2 - r(u+\bar{u}) + r^2$$
$$\leq 1 + r^2|u|^2 - r(u+\bar{u}) = (1-ru)(1-r\bar{u})$$

($\because$ if $0 < a \leq 1$, $0 < b \leq 1$, then

$$(1-a)(1-b) \geq 0$$

which is $1+ab \geq a+b$, equality hold if $a=1$ or $b=1$.

then let $a = r^2$, $b = |u|^2$.)

This says $|u-r|^2 \leq |1-r\bar{u}|^2$

with equality holds if $r=1$ or $|u|=1$.

(b) easy to check all.

$$F \circ F(z) = F\left(\frac{w-z}{1-\bar{w}z}\right) = \frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}} = \frac{(1-\bar{w}z)w - (w-z)}{1-\bar{w}z - \bar{w}(w-z)}$$

$$= \frac{w - |w|^2 z - w + z}{1-|w|^2} = \frac{z(1-|w|^2)}{1-|w|^2} = z$$

#8 let $h = h_1 + ih_2$, $g = g_1 + ig_2$, $f = f_1 + if_2$

$$\frac{\partial h}{\partial z} = \frac{\partial}{\partial z}(g \circ f) = \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial f_1} + \frac{\partial f_2}{\partial z}\frac{\partial g}{\partial f_2}$$

$$\frac{\partial f_1}{\partial z} = \frac{1}{2}\frac{\partial(f+\bar{f})}{\partial z} , \qquad \frac{\partial f_2}{\partial z} = \frac{1}{2i}\frac{\partial(f-\bar{f})}{\partial z}$$

$$\frac{\partial}{\partial f_1} = \frac{\partial f}{\partial f_1}\frac{\partial}{\partial f} + \frac{\partial \bar{f}}{\partial f_1}\frac{\partial}{\partial \bar{f}} = \left(\frac{\partial}{\partial f} + \frac{\partial}{\partial \bar{f}}\right)$$

$$\frac{\partial}{\partial f_2} = \frac{\partial f}{\partial f_2}\frac{\partial}{\partial f} + \frac{\partial \bar{f}}{\partial f_2}\frac{\partial}{\partial \bar{f}} = i\left(\frac{\partial}{\partial f} - \frac{\partial}{\partial \bar{f}}\right)$$

$$\left(\because f = f_1 + if_2, \qquad \bar{f} = f_1 - if_2\right)$$

$$\therefore \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial f_1} + \frac{\partial f_2}{\partial z}\frac{\partial g}{\partial f_2} = \frac{1}{2}\left(\frac{\partial f}{\partial z} + \frac{\partial \bar{f}}{\partial z}\right)\cdot\left(\frac{\partial g}{\partial f} + \frac{\partial g}{\partial \bar{f}}\right)$$
$$+ \frac{1}{2i}\left(\frac{\partial f}{\partial z} - \frac{\partial \bar{f}}{\partial z}\right) i \left(\frac{\partial g}{\partial f} - \frac{\partial g}{\partial \bar{f}}\right)$$

$$= \frac{1}{2}\left\{\frac{\partial f}{\partial z}\frac{\partial g}{\partial f} + \frac{\partial \bar{f}}{\partial z}\frac{\partial g}{\partial f} + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial \bar{f}} + \frac{\partial \bar{f}}{\partial z}\frac{\partial g}{\partial \bar{f}}\right.$$
$$\left. + \frac{\partial f}{\partial z}\frac{\partial g}{\partial f} - \frac{\partial \bar{f}}{\partial z}\frac{\partial g}{\partial f} - \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial \bar{f}} + \frac{\partial \bar{f}}{\partial z}\frac{\partial g}{\partial \bar{f}}\right\}$$

$$= \frac{\partial f}{\partial z}\frac{\partial g}{\partial f} + \frac{\partial \bar{f}}{\partial z}\frac{\partial g}{\partial \bar{f}} .$$

Conceptually :

$$\frac{\partial}{\partial z} = \left(\frac{\partial f_1}{\partial z}, \frac{\partial f_2}{\partial z}\right)\begin{pmatrix}\frac{\partial}{\partial f_1}\\ \frac{\partial}{\partial f_2}\end{pmatrix}$$

$$= \left(\frac{\partial f}{\partial z}, \frac{\partial \bar{f}}{\partial z} \right) \cdot A \cdot A^{-1} \begin{pmatrix} \frac{\partial}{\partial f} \\ \frac{\partial}{\partial \bar{f}} \end{pmatrix}$$

$$= \left(\frac{\partial f}{\partial z}, \frac{\partial \bar{f}}{\partial z} \right) \cdot \begin{pmatrix} \frac{\partial}{\partial f} \\ \frac{\partial}{\partial \bar{f}} \end{pmatrix}$$

where $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$