Last time : • what is a hol'c function

condition: $\forall z \in \Omega$, ($\Omega \subset \mathbb{C}$ open)

$$f'(z) := \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \quad \text{exists.}$$

• Equivalent Cauchy-Riemann condition

$$f(z) \text{ hol'c} \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $u = \mathrm{Re} f$, $v = \mathrm{Im} f$.

Today : ① watch youtube, Lec 5 of Borcherds's on example of hol'c function.

② power series.

A power series is $\sum_{n=0}^{\infty} a_n \cdot z^n$, $a_n \in \mathbb{C}$.

why we care ? • It is the most general way to write down a hol'c function near 0.

• How do we construct one ?

Review : • a series of numbers.

$$\sum_{n=0}^{\infty} C_n \qquad C_n \in \mathbb{C}.$$

we can ask whether it's convergent or not.

Def : $S_N = \sum_{n=0}^{N} C_n$. we say

$$\lim_{N \to \infty} S_N \text{ exist} \iff \sum_n C_n \text{ convergent.}$$

↑ partial sum.

<u>Cauchy criteria</u>:

$\sum_n C_n$ converges $\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists N$, s.t. $\forall\, N < n < m$, we have $\left| \sum_{j=n}^{m} c_j \right| < \varepsilon$.

- Necessary condition (sanity check).
  ΣC_n converges $\Rightarrow$ $\lim_{n\to\infty} C_n = 0$.

- <u>Tests</u> (sufficient condition)
  - ratio test : if $\limsup \left|\frac{C_{n+1}}{C_n}\right| = \rho < 1$, then ΣC_n converge.
  - root test : if $\limsup |C_n|^{\frac{1}{n}} = \rho < 1$, then ✓.

- Notion of absolute convergence :
  we say ΣC_n absolutely converges if $\Sigma |C_n|$ converge.

<u>Ex</u>:

- $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ $\begin{cases} \text{converges for } \alpha > 1 \\ \text{diverges for } 0 < \alpha \leq 1 \end{cases}$

- $\sum_{n=0}^{\infty} \rho^n$ $\begin{cases} \text{converges for } |\rho| < 1 \\ \text{diverges for } |\rho| \geq 1. \end{cases}$

Now, we consider $\sum_{n=0}^{\infty} a_n z^n$, we can ask, for what value of z, this converges.

Prop: There exist $\infty \geq R \geq 0$, such that

- if $|z| < R$, then $\sum a_n z^n$ is abs. convergent
- if $|z| > R$, then ——— is divergent.

more precisely, $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$

Pf: Let $L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

If $L = 0$, then $R = \infty$, then for any $z \in \mathbb{C}$,

$$\limsup |a_n \cdot z^n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} \cdot |z| = 0 \cdot |z| = 0$$

hence $\sum_n a_n z^n$ is abs. convergent.

If $L = \infty$. then $R = 0$, then for any $|z| > 0$,
$\sum a_n z^n$ is divergent by root test.

If $L \neq 0, \infty$. let $R = \frac{1}{L} \in (0, \infty)$.

(a) Need to show, $\forall |z| < R$, we have $\sum |a_n| \cdot |z|^n$ convergent.

$\because$ $|z| \cdot L < 1$

$\therefore$ $\exists \varepsilon > 0$, s.t. $|z| \cdot (L + \varepsilon) < 1$

since $\limsup |a_n|^{\frac{1}{n}} = L$.

for N large enough, we have $\forall n > N$.

$|a_n|^{\frac{1}{n}} < L + \varepsilon$.

Thus, $\sum_{n=N+1}^{\infty} |a_n| \cdot |z|^n < \sum_{n=N+1}^{\infty} (L+\varepsilon)^n \cdot |z|^n$

$= \sum_{n=N+1}^{\infty} [\underbrace{|z|(L+\varepsilon)}_{<1}]^n$. convergent. $<\infty$.

(b) If $|z| > R$, then $|z| \cdot L > 1$, $\exists\, \varepsilon > 0$, s.t. $|z| \cdot (L-\varepsilon) > 1$.

$\because$ $\limsup |a_n|^{\frac{1}{n}} = L$

$\therefore$ $\forall \exists N$, $\forall n > N$, we have a subseq of a_n. $\{a_{n_k}\}$.

s.t.

$$|a_{n_k}|^{\frac{1}{n_k}} > L - \varepsilon.$$

$$|a_{n_k}| \cdot |z|^{n_k} > (L-\varepsilon)^{n_k} \cdot |z|^{n_k} > 1.$$

so $\lim_{n\to\infty} |a_n| \cdot z^n$ is not going to be zero.

hence $\sum a_n \cdot z^n$ diverge. #.

Ex: • computation of radius of convergence.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n. \quad \text{(when is this valid?)}$$

Ans: valid for $|z| < 1$.

$\because$ $a_n = 1$ $\forall n$.

$\therefore$ $\limsup_n |a_n|^{\frac{1}{n}} = 1. = \frac{1}{R}$. $\Rightarrow R = 1$

by the prop. $\forall\, |z| < R$, it is abs convergent.

On $|z| = 1$, $\sum z^n$ is divergent.

($\because \lim z^n \neq 0$).