1. Power series defines hol'c function.

- Adding 2 hol'c functions, we still get a hol'c function.

- But, what if we add ∞ many hol'c function.

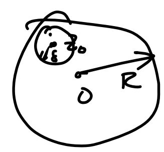
(Stein Thm 2.6)

<u>Thm</u>: Suppose $f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n$ has a radius of convergence R, then $f(z)$ is hol'c in $\{|z| < R\}$, with

$$f'(z) = \underbrace{\sum_{n=0}^{\infty} a_n \cdot n \cdot z^{n-1}}_{g(z)} \quad \text{valid } \forall \ |z| < R$$

This is not automatic, we don't always get

$$\frac{d}{dx} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{d}{dx} f_n(x)$$



<u>Pf</u>: Pick any $z_0 \in D_R(0)$.

Pick δ small enough, s.t. $|z_0| + \delta < R$.

Let $|h| < \delta$

$$f_N(z) = \sum_{n=0}^{N} a_n \cdot z^n$$

$$g_N(z) = \sum_{n=0}^{N} a_n \cdot n \cdot z^{n-1}$$

$$f_{>N}(z) = f(z) - f_N(z)$$

$$g_{>N}(z) = g - g_{>N}$$

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)$$

$$= \left(\frac{f_N(z_0+h) - f_N(z_0)}{h} - g_N(z_0) \right)$$

small as $h \to 0$.

$$+ \left(\underbrace{\frac{f_{>N}(z_0+h) - f_{>N}(z_0)}{h}}_{?} - g_{>N}(z_0) \right).$$

easily small $\because$ $g(z)$ is convergent series.

$$a^m - b^m = (a-b)(\underbrace{a^{m-1} + a^{m-2} \cdot b + a^{m-3} b^2 + \cdots + b^{m-1}}_{m \text{ terms.}})$$

$$\frac{(z_0+h)^m - z_0^m}{h.} = \frac{(z_0+h-z_0)((z_0+h)^{m-1} + \cdots + z_0^{m-1})}{h}$$

$$= (z_0+h)^{m-1} + \cdots + z_0^{m-1}.$$

$$\left|\frac{(z_0+h)^m - z_0^m}{h.}\right| \leq |(z_0+h)^{m-1}| + \cdots + |z_0^{m-1}|$$

$$\leq m \cdot (|z_0| + \delta)^{m-1}.$$

$$\left|\frac{f_{>N}(z_0+h) - f_{>N}(z_0)}{h}\right| = \left|\sum_{n=N+1}^{\infty} a_n \left(\frac{(z_0+h)^n - z_0^n}{h}\right)\right|$$

$$\leq \sum_{n=N+1}^{\infty} |a_n| \cdot \left|\frac{(z_0+h)^n - z_0^n}{h}\right|$$

$$\leq \sum_{n=N+1}^{\infty} \underbrace{|a_n| \cdot n \cdot (|z_0| + \delta)^{n-1}.}_{\text{this is a convergent series.}}$$

$$\limsup_{n\to\infty} [|a_n| \cdot n]^{\frac{1}{n}} = \underbrace{\left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.\right)}_{=\frac{1}{R}} \cdot \underbrace{\lim_{n\to\infty} n^{\frac{1}{n}}}_{=1.}$$

$\therefore$ by root test,

$$\sum_{n=N+1}^{\infty} |a_n| \cdot n \cdot (|z_0| + \delta)^{n-1} \quad \text{is convergent.}$$

Fix any $\varepsilon > 0$. We may pick N_1 large enough, such that $\sum_{n=N_1+1}^{\infty} |a_n| \cdot n \cdot (|z_0| + \delta)^{n-1} < \varepsilon$.

Similarly, since $\sum_{n=0}^{\infty} a_n \cdot n \cdot (z_0)^{n-1}$ is absolutely convergent, we can choose N_2 large enough, s.t.

$$\left| g_{>N_2}(z_0) \right| < \varepsilon.$$

Fix an $N > N_1$ and N_2.

Finally, since $\lim_{h \to 0} \left| \frac{f_N(z_0+h) - f_N(z_0)}{h} - g_N(z_0) \right|$

$= 0$

hence, $\exists \delta' > 0$ small enough, s.t. $\forall |h| < \delta'$,

$$\left| \frac{f_N(z_0+h) - f_N(z_0)}{h} - g_N(z_0) \right| < \varepsilon.$$

For all $|h| < \delta'$,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z) \right| < 3\varepsilon.$$

This shows,

$$\lim_{h \to 0} \left| \frac{f(z_0+h) - f(z_0)}{h} - g(z) \right| = 0. \qquad \#.$$

Later, we will show, if $f(z)$ is hol'c in a nbhd of z_0, then $f(z)$ admits Taylor expansion.

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

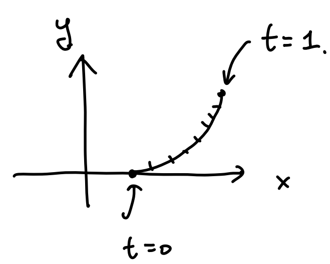
(Stein 1.3) Contour Integral.

- Recall line integral. : $\int_\gamma F\,dx + G\,dy.$

Let $(x(t), y(t))$ for $t \in [0,1]$ be a parametrized curve in $\mathbb{R}^2$

Ex: $\begin{cases} x(t) = 1+t \\ y(t) = t^2 \end{cases}$

t



Let $F(x,y)$, $G(x,y)$ be smooth functions on $\mathbb{R}^2$.

$$\int_0^1 F(x(t),y(t)) \cdot d\,x(t) + G(x(t),y(t))\, d\,y(t).$$

$$= \int_0^1 F(x(t),y(t)) \cdot \frac{d\,x(t)}{dt} \cdot dt + \int_0^1 G \cdot \frac{dy}{dt} \cdot dt.$$

$$= \int_0^1 \left(F \cdot \frac{dx}{dt} + G \frac{dy}{dt}\right) \cdot dt.$$

$$=: \int_\gamma F\,dx + G\,dy.$$

$$\gamma : [0,1] \to \mathbb{R}^2$$
$$t \mapsto (x(t), y(t)).$$

Ex (cont.).

$$F(x,y) = x^2$$
$$G(x,y) = x+y.$$

$x(t) = 1+t$

$y(t) = t^2.$

$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t.$

$$\int_\gamma F\,dx + G\,dy = \int_0^1 \left[(x(t))^2 \frac{dx}{dt} + (x(t)+y(t)) \cdot \frac{dy}{dt}\right] dt$$

$$= \int_0^1 \left[(1+t)^2 \cdot 1 + (1+t+t^2) \cdot 2t\right] \cdot dt.$$

$$= \cdots$$

any smooth function, not necessarily hol'c.

- Let $f(z) = u(x,y) + i\,V(x,y)$ $\qquad z = x+iy.$

$$dz = d(x+iy) = dx + i\,dy.$$

$$\gamma : [0,1] \to \mathbb{C}$$
$$t \mapsto z(t) = x(t) + i\,y(t)$$

$$\int_\gamma f(z)\,dz := \int (u+iv)\, d(x+iy)$$
$$= \int_\gamma u\cdot dx - v\,dy$$
$$+ i\cdot \int_\gamma u\,dy + v\cdot dx$$

well defined.

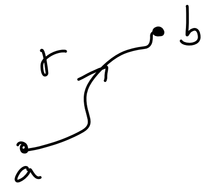
Ex: $f(z) = x := \mathrm{Re}(z)$.

$\gamma(t) = (x=1,\ y=t^2)$.

$$\int_\gamma f(z)\cdot dz = \int x(t)\cdot d(x(t)+i\,y(t))$$
$$= \int_0^1 1\cdot\ d(i\cdot t^2)$$
$$= \int_0^1 1\cdot i\cdot 2t\,dt = i$$

What if $f(z) = F'(z)$, where $F(z)$ is hol'c ?

Prop: Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth curve with $\gamma(0) = a$, $\gamma(1) = b$. Let $f(z) = F'(z)$, for F hol'c.



Then : $\displaystyle\int_\gamma f(z)\cdot dz = F(b) - F(a).$

Pf : $$\int_\gamma f(z)\, dz = \int_0^1 f(z(t))\, \frac{dz(t)}{dt} \cdot dt$$

$$= \int_0^1 \frac{dF}{dz} \cdot \frac{dz}{dt} \cdot dt.$$

$$= \int_0^1 \frac{d\,F(z(t))}{dt} \cdot dt.$$

$$= F(z(1)) - F(z(0)) = F(b) - F(a). \qquad \#.$$

Corollary : if $\gamma : [0,1] \to \mathbb{C}$ is a loop, i.e. $\gamma(0) = \gamma(1)$, and $f(z) = \frac{dF}{dz}$.

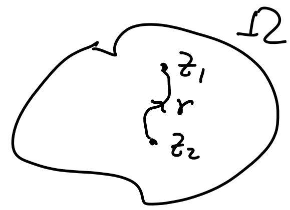
then $\displaystyle\int_\gamma f(z)\cdot dz = 0.$

Corollary : Say $f'(z) = 0 \quad \forall z \in \Omega$ ← connected domain in $\mathbb{C}$.

then $f(z) = c$ for some constant c.

Pf : $$f(z_1) - f(z_2) = \int_\gamma f'(z)\, dz$$

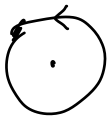
$$= \int_\gamma 0 \cdot dz = 0. \qquad \#.$$



Ex:

$$\oint_{|z|=1} \frac{1}{z} \cdot dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} \cdot \frac{d(e^{i\theta})}{d\theta} \cdot d\theta$$

$$= \int_0^{2\pi} \frac{1}{\cancel{e^{i\theta}}} \cdot i \cdot \cancel{e^{i\theta}} \cdot d\theta = i\underbrace{\int_0^{2\pi} d\theta}_{=2\pi i}$$



$|z|=1$

$z = e^{i\theta}$ θ goes from 0 to 2π.

counter clockwise.

$$\oint_{|z|=r} \cdot \frac{1}{z} \cdot dz = ?$$