Math 185. Midterm 2 2023.3.24.

- $\mathbb{D} = \{|z|<1\}$
- hol'c = holomorphic.

1. What's the definition of an open mapping?
(5pt) Give an example of a function $f: \mathbb{C} \to \mathbb{C}$ that is not an open mapping.

2. Let $f(z) = \dfrac{(z-1)}{(z-2)^2}$. Compute
(5pt)
$$\frac{1}{2\pi i} \oint_{|z|=3} \frac{f'(z)}{f(z)} \, dz$$
(Hint: Argument principle).

3. Let $f: \mathbb{D} \to \mathbb{C}$ be a hol'c function, and f is not
(5pts) a constant. Is it true that $\mathrm{Re}(f)$ cannot achieve maximum on $\mathbb{D}$? Explain.

4. Suppose you know that
(5pt)
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} e^{-2\pi i x \cdot \xi} \, dx = \pi \cdot e^{-2\pi|\xi|}$$
for all $\xi \in \mathbb{R}$.

Compute the Fourier transform for $f(x) = \dfrac{e^{2\pi i \cdot x}}{1+x^2}$.

5. Is it true that, if $|\hat{f}(\xi)| \leq C \cdot e^{-2\pi M|\xi|}$ for some $M>0$, then
(5pt) $f(x) := \int_{-\infty}^{+\infty} \hat{f}(\xi) \, e^{2\pi i \cdot x \cdot \xi} \, d\xi$ admits an analytic continuation to $S_M = \{ |\mathrm{Im}(z)| < M \}$? Please explain.

6. If we know $\hat{f}(\xi)$ vanishes for $\xi \leq 0$, and
(5pts)

$$\left|\hat{f}(\xi)\right| \leq \frac{A}{1+\xi^2} \qquad \forall\ \xi \in \mathbb{R}$$

Is it true that $f(x)$ admits an analytic continuation to the entire upper half plane $\overline{\mathbb{H}} = \{ \operatorname{Im}(z) \geq 0 \}$. ?
Please explain your reasoning briefly.

7. Suppose f is a holomorphic function on the annulus
(5pt)

$$r_1 < |z| < r_2$$

which of the integrals are independent of r?

$$I_1(r) := \int_0^{2\pi} \operatorname{Re}(f(re^{i\theta}))\, d\theta$$

$$I_2(r) := \int_0^{2\pi} |f(re^{i\theta})|\, d\theta$$

please explain briefly.

8. Show that $\prod_{n=1}^{\infty} \left(1+\frac{z}{n^2}\right)$ is convergent
(5pt)

$$\left(\forall x>0 \quad \Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt \right)$$

9. Express the following integral using Γ function:
(5pt).

$$\int_0^{\infty} e^{-x^3}\, dx$$

(hint: let $t = x^3$) then $x = t^{\frac{1}{3}}$.

10. What's the value of $\Gamma(-\frac{1}{2})$?
(5pt).

(you may use $\Gamma(\frac{1}{2}) = \sqrt{\pi}$)

<u>Solution</u> :

1. $f: X \to Y$ is an open map if for every open subset $U \subset X$. $f(U)$ is open in Y.

Non-example : $f(z) = 0 \quad \forall z \in \mathbb{C}$. constant map

2. Argument principle

$$\frac{1}{2\pi i} \int_{|z|=3} \frac{f'}{f} \, dz = \# \text{ of zero} - \# \text{poles} = 1-2 = -1$$

3. True. By maximum principle.

4.
$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) \cdot e^{-2\pi i \cdot x \cdot \xi} \, dx$$
$$= \int \frac{e^{2\pi i \cdot x}}{1+x^2} \, e^{-2\pi i \cdot x \cdot \xi} \, dx$$
$$= \int \frac{e^{-2\pi i \cdot x \cdot (\xi - 1)}}{1+x^2} \, dx$$
$$= \pi \cdot e^{-2\pi |\xi - 1|}$$

5. If. $|y| < M$, we have

$$|f(x+iy)| = \left| \int_{-\infty}^{+\infty} \hat{f}(\xi) \cdot e^{2\pi i \cdot (x+iy)\xi} \, d\xi \right|$$

$$\leq \int_{-\infty}^{+\infty} e^{-2\pi M |\xi|} \cdot e^{2\pi |y| \cdot |\xi|} \, d\xi$$

$$\leq \int_{-\infty}^{+\infty} e^{-2\pi (M - |y|) \cdot |\xi|} \, d\xi$$

$$< \infty$$

6. Yes. For $z = x+iy$, $y \geq 0$, we have

$$\left| \int_{-\infty}^{+\infty} e^{2\pi i (x+iy)\xi} \cdot \hat{f}(\xi) \, d\xi \right|$$

$$= \left| \int_{0}^{\infty} e^{2\pi i (x+iy)\xi} \hat{f}(\xi) \, d\xi \right|$$

$$= \int_{0}^{\infty} e^{-2\pi y \cdot \xi} |\hat{f}(\xi)| \, d\xi.$$

$$\leq \int_{0}^{\infty} 1 \cdot \frac{A}{1+\xi^2} \, d\xi < \infty$$

7. Yes.

$$I(r) = \mathrm{Re}\left(\int_0^{2\pi} f(re^{i\theta})\, d\theta \right)$$

$$= \mathrm{Re}\left(\oint_{|z|=r} f(z) \frac{dz}{iz} \right)$$

The contour integral is indep of r, as r varies between r_1, r_2 hence its real part is const.

8. Yes. Since

$$\sum_{n=1}^{\infty} \left| \frac{z}{n^2} \right| < \infty$$

9. $\int_0^\infty e^{-x^3} dx = \int_0^\infty e^{-t}\, d t^{\frac{1}{3}}$

$$= \int_0^\infty e^{-t}\, t^{\frac{1}{3}-1} \frac{1}{3}\, dt = \frac{1}{3} \Gamma\left(\frac{1}{3}\right)$$

10. $\Gamma\left(-\frac{1}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = \dfrac{\sqrt{\pi}}{(-1/2)}$