1.

(a)

$z_1 = z_2$**:** the whole plane, $\mathbb{C}$.

$z_1 \neq z_2$**:** the line $\frac{z_1+z_2}{2} + \mathbb{R}i(z_1 - z_2)$, which perpendicularly bisects the line that joins z_1 and z_2.

Proof:

Any $z \in \mathbb{C}$ can be expressed as

$$\begin{aligned} z &= \frac{z_1 + z_2}{2} + (s + it)(z_1 - z_2) \\ &= z_1 + ((s - \tfrac{1}{2}) + it)(z_1 - z_2) \\ &= z_2 + ((s + \tfrac{1}{2}) + it)(z_1 - z_2) \end{aligned}$$

giving

$$\begin{aligned} |z - z_1| &= |(s - \tfrac{1}{2}) + it||z_1 - z_2| \\ |z - z_2| &= |(s + \tfrac{1}{2}) + it||z_1 - z_2| \end{aligned}$$

$$|z - z_1| = |z - z_1| \iff s = 0$$

which proves the result.

(b) The unit circle.

$$|z| = 1 \implies \frac{1}{z} = \frac{1^2}{z} = \frac{z\overline{z}}{z} = \overline{z}$$

$$\frac{1}{z} = \overline{z} \implies |z| = \sqrt{z\overline{z}} = \sqrt{1} = 1$$

(c) The vertical line intersecting the real axis at 3.

(d) The half-plane strictly (resp. not strictly) to the right of the line $\mathrm{Re}(z) = c$.

(e)

$a \neq 0$**:** The result of rotating the half-plane $\mathrm{Re}(z) > -\frac{\mathrm{Re}(b)}{a}$ by the angle $-\arg(a)$.

$a = 0$**:**

$\mathrm{Re}(b) > 0$**:** the whole plane, $\mathbb{C}$.

$\mathrm{Re}(b) \leq 0$**:** nothing, $\emptyset$.

(f) The parabola $x = \frac{1}{2}(y^2 - 1)$

Proof:

$$\begin{aligned} |z| = x + 1 &\implies |z| = |x + 1| \\ &\iff |z|^2 = |x + 1|^2 \\ &\iff x^2 + y^2 = x^2 + 2x + 1 \\ &\iff y^2 = 2x + 1 \\ &\iff x = \tfrac{1}{2}(y^2 - 1) \end{aligned}$$

The above gives

$$\begin{aligned} x = \tfrac{1}{2}(y^2 - 1) &\implies |z| = |x + 1| \text{ and } x \geq -\tfrac{1}{2} \\ &\implies |z| = |x + 1| \text{ and } x + 1 \geq 0 \\ &\implies |z| = x + 1 \end{aligned}$$

(g) The horizontal line intersecting the imaginary axis at ci.

2.

$$
\begin{aligned}
\langle z, w\rangle &= x_1x_2 + y_1y_2 \\
&= \tfrac{1}{2}[2x_1x_2 + 2y_1y_2] \\
&= \tfrac{1}{2}[x_1x_2 - ix_1y_2 + iy_1x_2 + y_1y_2 + x_2x_1 - ix_2y_1 + iy_2x_1 + y_2y_1] \\
&= \tfrac{1}{2}[(x_1 + iy_1)(x_2 - iy_2) + (x_2 + iy_2)(x_1 - iy_1)] \\
&= \tfrac{1}{2}[z\overline{w} + w\overline{z}] \\
&= \tfrac{1}{2}[(z, w) + (w, z)] \\
&= \tfrac{1}{2}[(z, w) + \overline{(z, w)}] \\
&= \tfrac{1}{2}[2\,\mathrm{Re}(z, w)] \\
&= \mathrm{Re}(z, w)
\end{aligned}
$$

3.

$s = 0$: one solution, $z = 0$.

$s > 0$: n solutions, written below.

Write $z = re^{i\theta}$. We have

$$\begin{aligned} z^n = \omega \quad &\iff \quad r^n e^{in\theta} = se^{i\varphi} \\ &\iff \quad r^n = s \text{ and } e^{in\theta} e^{i\varphi} \\ &\iff \quad r = s^{\frac{1}{n}} \text{ and } \quad n\theta \equiv \varphi \pmod{2\pi} \\ &\iff \quad r = s^{\frac{1}{n}} \text{ and } \quad n\theta = \varphi + 2\pi k \text{ for some } k \in \mathbb{Z} \\ &\iff \quad r = s^{\frac{1}{n}} \text{ and } \quad \theta = \frac{\varphi}{n} + 2\pi \frac{k}{n} \text{ for some } k \in \mathbb{Z} \end{aligned}$$

This shows that every solution is equal to one of

$$s^{\frac{1}{n}} e^{i\frac{\varphi}{n}}, \; s^{\frac{1}{n}} e^{i(\frac{\varphi}{n} + 2\pi\frac{1}{n})}, \; \ldots \; s^{\frac{1}{n}} e^{i(\frac{\varphi}{n} + 2\pi\frac{n-1}{n})}$$

All of these are distinct, hence there are exactly n solutions.

7.

(a)

Let $z, w \in \mathbb{C}$ with $\overline{w}z \neq 1$.
Write $z = x + iy$ and $w = u + iv$.

Then each of the following inequalities
holds strictly if $|z| < 1$ and $|w| < 1$,
and holds as equality if $|z| = 1$ or $|w| = 1$.

$$\begin{aligned}
0 &\leq [(u^2+v^2)-1][(x^2+y^2)-1] \\
0 &\leq (u^2+v^2)(x^2+y^2) - (u^2+v^2) - (x^2+y^2) + 1 \\
(u^2+v^2) + (x^2+y^2) &\leq 1 + (u^2+v^2)(x^2+y^2) \\
u^2+v^2+x^2+y^2 &\leq 1 + u^2x^2 + u^2y^2 + v^2x^2 + v^2y^2 \\
u^2+v^2+x^2+y^2 &\leq 1 + (u^2x^2 + v^2y^2 + 2uxvy) + (u^2y^2 + v^2x^2 - 2uxvy) \\
u^2+v^2+x^2+y^2 &\leq 1 + (ux+vy)^2 + (uy-vx)^2 \\
u^2+v^2+x^2+y^2 - 2ux - 2vy &\leq 1 - 2(ux+vy) + (ux+vy)^2 + (uy-vx)^2 \\
(u-x)^2 + (v-y)^2 &\leq [1-(ux+vy)]^2 + (uy-vx)^2 \\
\Big|(u-x) + i(v-y)\Big|^2 &\leq \Big|[1-(ux+vy)] - i(uy-vx)\Big|^2 \\
\Big|w - z\Big|^2 &\leq \Big|1 - \overline{w}z\Big|^2 \\
\Big|w - z\Big| &\leq \Big|1 - \overline{w}z\Big| \\
\left|\frac{w-z}{1-\overline{w}z}\right| &\leq 1
\end{aligned}$$

(b)

(i)

Part (a) shows that $F(\mathbb{D}) \subseteq \mathbb{D}$.
F is holomorphic on $\mathbb{D}$ because:

- We know $z \mapsto z$ is entire,
- so $z \mapsto w - z$ and $z \mapsto 1 - \overline{w}z$ are entire. The latter is nonzero on $\mathbb{D}$,
- so $z \mapsto \frac{1}{1-\overline{w}z}$ is holomorphic on $\mathbb{D}$,
- so $z \mapsto \frac{w-z}{1-\overline{w}z}$ is holomorphic on $\mathbb{D}$.

(ii)

$$\begin{aligned}
F(0) &= \frac{w-0}{1-\overline{w}0} \\
&= \frac{w}{1} \\
&= w
\end{aligned}$$

$$\begin{aligned}
F(w) &= \frac{w-w}{1-\overline{w}w} \\
&= \frac{0}{1-\overline{w}w} \\
&= 0
\end{aligned}$$

(iii) Follows from part (a).

(iv)

For any $z \in \mathbb{D}$:

$$\begin{aligned}(F \circ F)(z) &= \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \overline{w}\frac{w-z}{1-\overline{w}z}} \\ &= \frac{w(1-\overline{w}z) - (w-z)}{(1-\overline{w}z) - \overline{w}(w-z)} \\ &= \frac{w - w\overline{w}z - w + z}{1 - \overline{w}z - \overline{w}w + \overline{w}z} \\ &= \frac{z(1-w\overline{w})}{1-\overline{w}w} \\ &= z\end{aligned}$$

So $F \circ F : \mathbb{D} \to \mathbb{D}$ equals $\mathrm{id}_{\mathbb{D}}$, so $F : \mathbb{D} \to \mathbb{D}$ is bijective.

8.

At $z_0 \in U$, we have

$$
\begin{aligned}
\frac{\partial h}{\partial z}(z_0) &= \frac{1}{2}\left(\frac{\partial h}{\partial x}(z_0) + \frac{1}{i}\frac{\partial h}{\partial y}(z_0)\right) \\
&= \frac{1}{2}\left((Dg)_{f(z_0)}\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}(Dg)_{f(z_0)}\frac{\partial f}{\partial y}(z_0)\right) \\
&= \frac{1}{2}\Bigg(\quad \left(\frac{\partial g}{\partial x}(f(z_0))\frac{\partial \operatorname{Re} f}{\partial x}(z_0) \quad + \quad \frac{\partial g}{\partial y}(f(z_0))\frac{\partial \operatorname{Im} f}{\partial x}(z_0)\right) \\
&\qquad + \frac{1}{i}\left(\frac{\partial g}{\partial x}(f(z_0))\frac{\partial \operatorname{Re} f}{\partial y}(z_0) \quad + \quad \frac{\partial g}{\partial y}(f(z_0))\frac{\partial \operatorname{Im} f}{\partial y}(z_0)\right)\Bigg) \\
&= \frac{\partial g}{\partial x}(f(z_0))\frac{\partial \operatorname{Re} f}{\partial z}(z_0) \quad + \quad \frac{\partial g}{\partial y}(f(z_0))\frac{\partial \operatorname{Im} f}{\partial z}(z_0) \\
&= \quad \frac{1}{2}\frac{\partial g}{\partial x}(f(z_0))\ 2\frac{\partial \operatorname{Re} f}{\partial z}(z_0) \\
&\quad + \frac{1}{2}\frac{\partial g}{\partial y}(f(z_0))\ 2\frac{\partial \operatorname{Im} f}{\partial z}(z_0) \\
&= \quad \frac{1}{2}\frac{\partial g}{\partial x}(f(z_0))\left(\left(\frac{1}{2}\frac{\partial f}{\partial x}(z_0) + \frac{1}{2}\frac{\partial \overline{f}}{\partial x}(z_0)\right) + \frac{1}{i}\left(\frac{1}{2}\frac{\partial f}{\partial y}(z_0) + \frac{1}{2}\frac{\partial \overline{f}}{\partial y}(z_0)\right)\right) \\
&\quad + \frac{1}{2}\frac{\partial g}{\partial y}(f(z_0))\left(\left(\frac{1}{2}\frac{\partial f}{\partial x}(z_0) - \frac{1}{2}\frac{\partial \overline{f}}{\partial x}(z_0)\right) + \frac{1}{i}\left(\frac{1}{2}\frac{\partial f}{\partial y}(z_0) - \frac{1}{2}\frac{\partial \overline{f}}{\partial y}(z_0)\right)\right) \\
&= \quad \frac{1}{2}\left(\frac{\partial g}{\partial x}(f(z_0)) + \frac{1}{i}\frac{\partial g}{\partial y}(f(z_0))\right)\frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) \\
&\quad + \frac{1}{2}\left(\frac{\partial g}{\partial x}(f(z_0)) - \frac{1}{i}\frac{\partial g}{\partial y}(f(z_0))\right)\frac{1}{2}\left(\frac{\partial \overline{f}}{\partial x}(z_0) + \frac{1}{i}\frac{\partial \overline{f}}{\partial y}(z_0)\right) \\
&= \frac{\partial g}{\partial z}(f(z_0))\frac{\partial f}{\partial z}(z_0) \quad + \quad \frac{\partial g}{\partial \overline{z}}(f(z_0))\frac{\partial \overline{f}}{\partial z}(z_0)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial h}{\partial \overline{z}}(z_0) &= \frac{1}{2}\left(\frac{\partial h}{\partial x}(z_0) - \frac{1}{i}\frac{\partial h}{\partial y}(z_0)\right) \\
&= \frac{1}{2}\left((Dg)_{f(z_0)}\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}(Dg)_{f(z_0)}\frac{\partial f}{\partial y}(z_0)\right) \\
&= \frac{1}{2}\Bigg(\quad \left(\frac{\partial g}{\partial x}(f(z_0))\frac{\partial \operatorname{Re} f}{\partial x}(z_0) \quad + \quad \frac{\partial g}{\partial y}(f(z_0))\frac{\partial \operatorname{Im} f}{\partial x}(z_0)\right) \\
&\qquad - \frac{1}{i}\left(\frac{\partial g}{\partial x}(f(z_0))\frac{\partial \operatorname{Re} f}{\partial y}(z_0) \quad + \quad \frac{\partial g}{\partial y}(f(z_0))\frac{\partial \operatorname{Im} f}{\partial y}(z_0)\right)\Bigg) \\
&= \frac{\partial g}{\partial x}(f(z_0))\frac{\partial \operatorname{Re} f}{\partial \overline{z}}(z_0) \quad + \quad \frac{\partial g}{\partial y}(f(z_0))\frac{\partial \operatorname{Im} f}{\partial \overline{z}}(z_0) \\
&= \quad \frac{1}{2}\frac{\partial g}{\partial x}(f(z_0))\ 2\frac{\partial \operatorname{Re} f}{\partial \overline{z}}(z_0) \\
&\quad + \frac{1}{2}\frac{\partial g}{\partial y}(f(z_0))\ 2\frac{\partial \operatorname{Im} f}{\partial \overline{z}}(z_0) \\
&= \quad \frac{1}{2}\frac{\partial g}{\partial x}(f(z_0))\left(\left(\frac{1}{2}\frac{\partial f}{\partial x}(z_0) + \frac{1}{2}\frac{\partial \overline{f}}{\partial x}(z_0)\right) - \frac{1}{i}\left(\frac{1}{2}\frac{\partial f}{\partial y}(z_0) + \frac{1}{2}\frac{\partial \overline{f}}{\partial y}(z_0)\right)\right) \\
&\quad + \frac{1}{2}\frac{\partial g}{\partial y}(f(z_0))\left(\left(\frac{1}{2}\frac{\partial f}{\partial x}(z_0) - \frac{1}{2}\frac{\partial \overline{f}}{\partial x}(z_0)\right) - \frac{1}{i}\left(\frac{1}{2}\frac{\partial f}{\partial y}(z_0) - \frac{1}{2}\frac{\partial \overline{f}}{\partial y}(z_0)\right)\right) \\
&= \quad \frac{1}{2}\left(\frac{\partial g}{\partial x}(f(z_0)) + \frac{1}{i}\frac{\partial g}{\partial y}(f(z_0))\right)\frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) \\
&\quad + \frac{1}{2}\left(\frac{\partial g}{\partial x}(f(z_0)) - \frac{1}{i}\frac{\partial g}{\partial y}(f(z_0))\right)\frac{1}{2}\left(\frac{\partial \overline{f}}{\partial x}(z_0) - \frac{1}{i}\frac{\partial \overline{f}}{\partial y}(z_0)\right) \\
&= \frac{\partial g}{\partial z}(f(z_0))\frac{\partial f}{\partial \overline{z}}(z_0) \quad + \quad \frac{\partial g}{\partial \overline{z}}(f(z_0))\frac{\partial \overline{f}}{\partial \overline{z}}(z_0)
\end{aligned}
$$