9.

Intro.

We examine

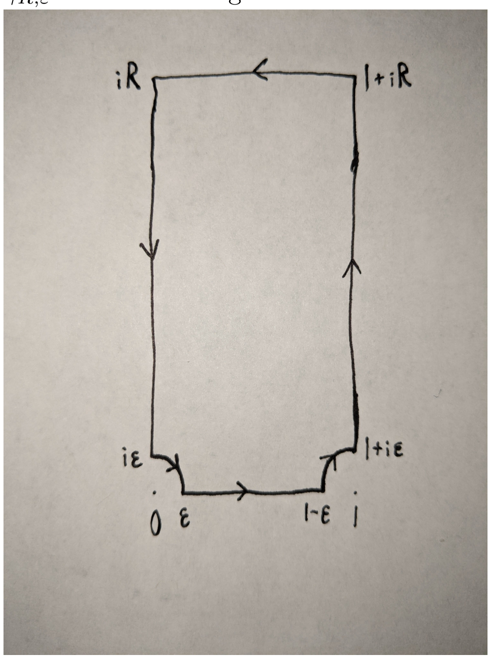
$$\int_{\gamma_{R,\varepsilon}} \log(\sin \pi z) dz$$

where

- log is defined on $\mathbb{C} \setminus (-\infty, 0]$ by

$$\log(re^{i\theta}) = \log r + i\theta \quad \theta \in (-\pi, \pi)$$

- $\gamma_{R,\varepsilon}$ is a tall rectangle indented with small quarter-circles:



The rectangle has width 1 and height R.
The quarter-circles have radius ε.

Notation.

For convenience,

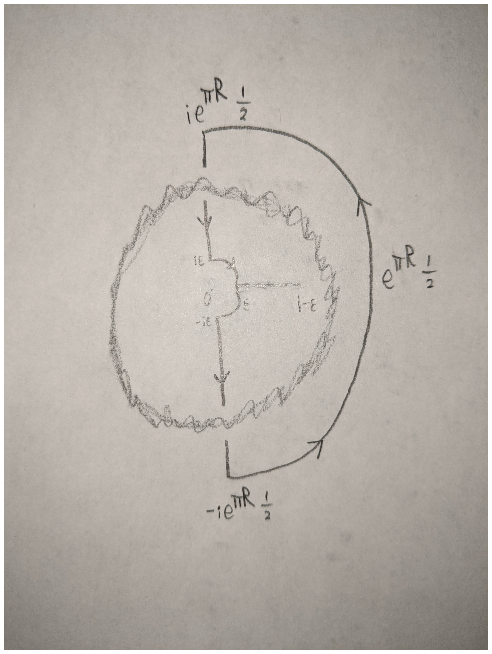
$$\gamma = \gamma_{R,\varepsilon}$$

Path of $\sin \pi z$.

We examine the path of $\sin \pi z$ as z runs over each piece of γ.
Two purposes:

- To verify that $\sin \pi z$ stays in the domain of our logarithm.
- To help evaluate $\log(\sin \pi z)$ later.

Below is a drawing of the path.
Squiggles indicate a change of scale.



Vertical sides:

Left side: $z = iy$, $\varepsilon \le y \le R$

Positive imaginary axis,

i.e. $\sin \pi z = ic$ with $c > 0$.

$$\begin{aligned}
\sin \pi z &= \tfrac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right) \\
&= \tfrac{1}{2i}(\underbrace{e^{-\pi y} - e^{\pi y}}_{<0}) \\
&= i\,\underbrace{\tfrac{1}{2}(e^{\pi y} - e^{-\pi y})}_{>0}
\end{aligned}$$

Right side: $z = 1 + iy$, $\varepsilon \le y \le R$

Negative imaginary axis,

i.e. $\sin \pi z = -ic$ with $c > 0$.

$$\begin{aligned}
\sin \pi z &= \tfrac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right) \\
&= \tfrac{1}{2i}(e^{i\pi}e^{-\pi y} - e^{-i\pi}e^{\pi y}) \\
&= \tfrac{1}{2i}(\underbrace{-e^{-\pi y} - -e^{\pi y}}_{>0}) \\
&= -i\,\underbrace{\tfrac{1}{2}(e^{\pi y} - e^{-\pi y})}_{>0}
\end{aligned}$$

Horizontal sides:

Bottom side: $z = x$, $\varepsilon \le x \le 1 - \varepsilon$

Positive real.

Top side: $z = x + iR$, $\varepsilon \le x \le 1 - \varepsilon$

Roughly a big semicircle centered at 0,
in right half-plane.
$\sin \pi z \approx e^{\pi R} e^{i\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\begin{aligned}\sin(\pi z) &= \frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right)\\ &= \frac{1}{2i}\left(e^{i\pi x}e^{-\pi R} - e^{-i\pi x}e^{\pi R}\right)\\ &= i\frac{1}{2}\left(e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}\right)\\ &\approx i\frac{1}{2}e^{-i\pi x}e^{\pi R}\end{aligned}$$

Note that, even before approximating, we have

$$\operatorname{Im}\left(e^{-i\pi x}e^{\pi R}\right) < \operatorname{Im}\left(e^{i\pi x}e^{-\pi R}\right)$$
$$\operatorname{Im}\left(e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}\right) < 0$$
$$\operatorname{Re}\sin \pi z > 0$$

for $0 < x < 1$,
which shows that the path stays in the domain of log.

Quarter circles:

Left quarter circle: $z = \varepsilon e^{i\theta}$, $\theta \in [0, \frac{\pi}{2}]$

$\sin \pi z$ travels roughly the same quarter circle as πz,
i.e. $\sin \pi z \approx \pi z$.
(This is in upper-right quadrant of the plane.)
Using the power series centered at 0:

$$\sin \pi z = \pi z + z^2 \phi(z)$$

where ϕ is holomorphic in a neighborhood of 0.

When $\varepsilon = |z|$ is sufficiently small,
at least one of

$$|\operatorname{Im}(\pi z)| > \left|\operatorname{Im}(z^2\phi(z))\right|$$
$$|\operatorname{Re}(\pi z)| > \left|\operatorname{Re}(z^2\phi(z))\right|$$

must hold, ensuring that $\sin \pi z$ is in the domain of log.
(Recall that πz lies in the upper-right quadrant.)

Note also that, when ε is sufficiently small,
our formula for $\sin \pi z$ gives

$$\varepsilon < |\sin \pi z| < 1$$

Right quarter circle: $z = 1 + \varepsilon e^{i\theta}$, $\theta \in [\frac{\pi}{2}, \pi]$

$\sin \pi z$ travels roughly the same quarter circle as $-\pi(z-1)$,
i.e. $\sin \pi z \approx -\pi(z-1)$.
(This is in lower-right quadrant of the plane.)
We also have the bound

$$\varepsilon < |\sin \pi z| < 1$$

when ε is sufficiently small.
Reasoning is similar to previous case, using the formula

$$\sin(\pi z) = -\pi(z-1) + (z-1)^2\psi(z)$$

Evaluating pieces.

Vertical.

$$\begin{aligned}
&\int_{y=R}^{\varepsilon} \log(\sin \pi(iy)) i dy + \int_{y=\varepsilon}^{R} \log(\sin \pi(1+iy)) i dy \\
&= \int_{y=\varepsilon}^{R} \big[\log(\sin \pi(1+iy)) - \log(\sin \pi(iy))\big] i dy \\
&= \int_{y=\varepsilon}^{R} \big[(-i\tfrac{\pi}{2} + \log\big|\tfrac{1}{2}(e^{\pi y} - e^{-\pi y})\big|) - (i\tfrac{\pi}{2} + \log\big|\tfrac{1}{2}(e^{\pi y} - e^{-\pi y})\big|)\big] i dy \\
&= \int_{y=\varepsilon}^{R} -i\pi i dy \\
&= \int_{y=\varepsilon}^{R} \pi dy \\
&= \pi(R-\varepsilon)
\end{aligned}$$

This approaches

$$\pi R$$

as $\varepsilon \to 0^+$.

Top.

$$\begin{aligned}
&\int_{x=1}^{0} \log(\sin \pi(x+iR)) dx \\
&= \int_{x=1}^{0} \big[\log \frac{\sin \pi(x+iR)}{|\sin \pi(x+iR)|} + \log|\sin \pi(x+iR)|\big] dx \\
&= \int_{x=1}^{0} \big[\log i \frac{e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}}{|e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}|} + \log \tfrac{1}{2}\left|e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}\right|\big] dx \\
&= \int_{x=1}^{0} \log i \frac{e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}}{|e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}|} dx \\
&\quad + \int_{x=1}^{0} \log \tfrac{1}{2} e^{\pi R} dx \\
&\quad + \int_{x=1}^{0} \log\left|e^{-i\pi x} - e^{i\pi x}e^{-2\pi R}\right| dx
\end{aligned}$$

As $R \to \infty$,

$$i\frac{e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}}{|e^{-i\pi x}e^{\pi R} - e^{i\pi x}e^{-\pi R}|} \to ie^{-i\pi x} = e^{i(-\pi x + \frac{\pi}{2})}$$

$$\left|e^{-i\pi x} - e^{i\pi x}e^{-2\pi R}\right| \to 1$$

uniformly in x (and note that $-\pi x + \frac{\pi}{2} \in (-\pi, \pi)$).
Due to the compactness of

$$\left\{e^{i(-\pi x + \frac{\pi}{2})} \;\middle|\; x \in [0,1]\right\}$$

$$\{1\}$$

we find that log is uniformly continuous on these sets,
hence log preserves the uniform convergence,
hence our sum of integrals approaches

$$
\begin{aligned}
&= \int_{x=1}^{0} i(-\pi x + \tfrac{\pi}{2})dx \\
&\quad + \int_{x=1}^{0} \big[-\log 2 + \pi R\big]dx \\
&\quad + \int_{x=1}^{0} 0dx \\
&= i(\tfrac{\pi}{2} - \tfrac{\pi}{2}) + (\log 2 - \pi R) + 0 \\
&= \log 2 - \pi R
\end{aligned}
$$

Bottom.

The integral

$$\int_{x=\varepsilon}^{1-\varepsilon} \log(\sin \pi x)dx$$

approaches

$$\int_{x=0}^{1} \log(\sin \pi x)dx$$

as $\varepsilon \to 0^+$.

Quarter circles.

We use the bound

$$\left|\int_{\text{quarter circle}} \log(\sin \pi z)dz\right| < M\tfrac{\pi}{2}\varepsilon$$

where M is the maximum magnitude of the integrand on the contour and $\frac{\pi}{2}\varepsilon$ is the length of the contour.

Recall that on the quarter circles we have

$$\varepsilon < |\sin \pi z| < 1$$

The log is

$$\log(\sin \pi z) = \mu(z) + \log|\sin \pi z|$$

where $\mu(z)$ is purely imaginary and $|\mu(z)| < \pi$.

$$\log \varepsilon < \log|\sin \pi z| < 0$$

Hence

$$|\log(\sin \pi z)| < \pi + |\log \varepsilon|$$

giving the bound

$$\left|\int_{\text{quarter circle}} \log(\sin \pi z)dz\right| < (\pi + |\log \varepsilon|)\tfrac{\pi}{2}\varepsilon$$

which approaches 0 as ε does.
(In particular, we use the fact that $\varepsilon \log \varepsilon \to 0$ as $\varepsilon \to 0^+$.)

Assembly, conclusion.

Immediately

$$0 = \int_\gamma \log(\sin \pi z) dz$$

because the integrand is holomorphic on the contour and its interior.

This integral equals the sum of the integrals over the pieces of γ;
as $R \to \infty$ and $\varepsilon \to 0^+$ together (e.g. letting $R = \frac{1}{\varepsilon}$ and having $R \to \infty$),
this sum of integrals remains approaches

$$\pi R + (\log 2 - \pi R) + \int_0^1 \log(\sin \pi x) dx$$

$$\log 2 + \int_0^1 \log(\sin \pi x) dx$$

It also approaches 0, since it is constantly 0.
Hence

$$0 = \log 2 + \int_0^1 \log(\sin \pi x) dx$$

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

QED.

12.

Poles.

The poles of f are at the integers and $-u$.

Residues.

$\text{res}_n f$**.**

To determine $\text{res}_n f$ for $n \in \mathbb{Z}$, we note

$$f(z) = (u+z)^{-2}\pi\cos\pi z \;\cdot\; \frac{1}{\sin\pi z}$$

From previous homework, we have

$$\frac{1}{\sin\pi z} = \frac{(-1)^n}{\pi}(z-n)^{-1} + F(z)$$

with F holomorphic on a neighborhood of n.
Also

$$(u+z)^{-2}\pi\cos\pi z = (u+n)^{-2}\pi(-1)^n + zG(z)$$

with G holomorphic on a neighborhood of n.
Hence

$$\frac{1}{\sin\pi z} = (u+n)^{-2}(z-n)^{-1} + H(z)$$

with H holomorphic on a neighborhood of n,
therefore

$$\text{res}_n f = (u+n)^{-2}$$

$\text{res}_{-u} f$**.**

Calculating 0-th and 1-st derivatives of $\pi\cot\pi z$ at $-u$,
we obtain

$$\pi\cot\pi z = -\pi\cot\pi u - \frac{\pi^2}{(\sin u)^2}(z+u) + z^2 I(z)$$

with I holomorphic on a neighborhood of $-u$. Hence

$$\begin{aligned}\frac{\pi\cot\pi z}{(u+z)^2} &= (z+u)^{-2}\left(-\pi\cot\pi u - \frac{\pi^2}{(\sin u)^2}(z+u) + z^2 I(z)\right)\\ &= -\pi(\cot\pi u)(z+u)^{-2} - \frac{\pi^2}{(\sin u)^2}(z+u)^{-1} + I(z)\end{aligned}$$

$$\text{res}_{-u} f = -\frac{\pi^2}{(\sin u)^2}$$

Linking integral to summation.

When N is large enough,
the residue formula gives

$$\tfrac{1}{2\pi i}\int_{|z|=R_N}^{+} f(z)dz \;= -\frac{\pi^2}{(\sin u)^2} + \sum_{n=-N}^{N}\frac{1}{(u+n)^2}$$

Decay of integral.

$|\cot\pi z| < 5$**.**

We prove the above result when $|z| = R_N$ and N is sufficiently large.

$$\begin{aligned}\cot z &= \frac{\cos z}{\sin z} \\ &= \frac{\frac{1}{2}\left(e^{iz} + e^{-iz}\right)}{\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)} \\ &= i\frac{e^{i2z} + 1}{e^{i2z} - 1} \\ &= i\left(1 + \frac{2}{e^{i2z} - 1}\right)\end{aligned}$$

$$\cot \pi z = i\left(1 + \frac{2}{e^{2\pi iz} - 1}\right)$$

When $\operatorname{Im}(z) > 1$ we have

$$\left|e^{2\pi iz}\right| < \frac{1}{e^{2\pi}}$$

$$\left|e^{2\pi iz} - 1\right| > 1 - \frac{1}{e^{2\pi}}$$

$$\begin{aligned}|\cot \pi z| &< 1 + \frac{2}{1 - \frac{1}{e^{2\pi}}} \\ &< 5\end{aligned}$$

When $\operatorname{Im}(z) < -1$ we have

$$\left|e^{2\pi iz}\right| > e^{2\pi}$$

$$\begin{aligned}|\cot \pi z| &< 1 + \frac{2}{e^{2\pi} - 1} \\ &< 5\end{aligned}$$

When $|\operatorname{Im}(z)| \leq 1$,
if N is big enough we have

$$R_N - |\operatorname{Re}(z)| < \tfrac{1}{4}$$

$$N + \tfrac{1}{4} < |\operatorname{Re}(z)| < N + \tfrac{3}{4}$$

$$\operatorname{Re}(e^{2\pi iz}) < 0$$

$$\operatorname{Re}(e^{2\pi iz} - 1) < -1$$

$$\left|e^{2\pi iz} - 1\right| > 1$$

$$\left|\frac{2}{e^{2\pi iz} - 1}\right| < 2$$

$$\begin{aligned}|\cot \pi z| &< 1 + 2 \\ &< 5\end{aligned}$$

Hence

$$\left| \int_{|z|=R_N}^{+} \frac{\pi \cot \pi z}{(u+z)^2} dz \right| \le (2\pi R_N) \frac{5\pi}{R_N^2 - |u|}$$

which approaches 0 as $N \to \infty$.

Conclusion.

The above sections show that

$$\begin{aligned} 0 &= \lim_{N\to\infty} \int_{|z|=R_N}^{+} \frac{\pi \cot \pi z}{(u+z)^2} dz \\ &= -\frac{\pi^2}{(\sin u)^2} + \sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} \end{aligned}$$

Hence the desired result.

16.

(a)

If ϵ is small enough,
then $|f| < |\epsilon g|$ on the unit circle,
hence $\underbrace{f + \epsilon g}_{=f_\epsilon}$
has no zeros on the unit circle
and has the same number of zeros in $\mathbb{D}$ as f does,
i.e. has exactly one.
Hence f_ϵ has a unique zero in $|z| \leq 1$.

(b)

By the proof of (a),
we know that if ϵ is sufficiently small
then f_ϵ has a unique zero on $|z| < 1$.

Let ϵ_0 be small enough for that.
We will prove the continuity of $\epsilon \mapsto z_\epsilon$ at ϵ_0.

Let $r > 0$ be small enough that $\overline{D_r}(z_{\epsilon_0}) \subseteq D_1(0)$.
Define $\delta = \frac{m}{M}$, where

$$m = \min_{z - z_{\epsilon_0}} |f_{\epsilon_0}|$$

$$M = \max_{z - z_{\epsilon_0}} |g|$$

Then whenever $|\epsilon - \epsilon_0| < \delta$, we have

$$|f_{\epsilon_0}| > |(\epsilon - \epsilon_0) g|$$

on the circle $|z - z_{\epsilon_0}| < r$.
So by Rouché's theorem,
the function $f_\epsilon = f_{\epsilon_0} + (\epsilon - \epsilon_0) g$
has a root on $D_r(z_{\epsilon_0})$ (since f_{ϵ_0} does),
so $|z_\epsilon - z_{\epsilon_0}| < r$.

Since $r > 0$ was an arbitrary small number,
we find that for any $r > 0$
there exists δ small enough that
$|\epsilon - \epsilon_0| < \delta \implies |z_\epsilon - z_{\epsilon_0}| < r$,
so $\epsilon \mapsto z_\epsilon$ is continuous.

#4

Zeros on $|z| < 2$: 4.

Proof:

If $|z| \not< 2$ then

$$\begin{aligned}\left|z^4 - 6z + 3\right| &\geq |z|^4 - |-6z| - |3| \\ &= |z|^4 - 6|z| - 3 \\ &= |z|(|z|^3 - 6) - 3 \\ &\geq 2(8-6) - 3 \\ &= 1 \\ &> 0\end{aligned}$$

hence all the roots are in $|z| < 2$,
and by fundamental theorem of algebra, there are 4.

Zeros on $|z| < 1$: 1.

Proof:

On $|z| = 1$,

$$\begin{aligned}\left|z^4 - 6z + 3\right| &\geq -|z|^4 + |-6z| - |3| \\ &= -1 + 6 - 3 \\ &= 2 \\ &> 1 \\ &= \left|-z^4\right|\end{aligned}$$

Hence the function

$$-6z + 3 = (z^4 - 6z + 3) + (-z^4)$$

has the same number of zeros on $|z| < 1$ as $z^4 - 6z + 3$.
This function has a unique zero $\frac{1}{2}$,
which lies in $|z| < 1$,
hence $z^4 - 6z + 3$ also has exactly one zero there.

#5

Note: to prove that a mapping is open,
it suffices to show that it maps open discs to open sets.
Indeed, this condition implies that the map takes open sets
(i.e. unions of open discs)
to unions of open sets (i.e. open sets).

$f : \mathbb{C} \to \mathbb{C}$**,** $f(z) = \overline{z}$**:** open.

Reason: f maps $D_r(z)$ to $D_r(\overline{z})$.
Indeed, if $|w - z| < r$
then $|\overline{w} - \overline{z}| = |\overline{w - z}| = |w - z| < r$.
(f is just reflection across horizontal axis.)

$f : \mathbb{C} \to \mathbb{R}$**,** $f(z) = |z|^2$**:** not open.

f maps $D_1(0)$ to $[0, 1)$.

$f : \mathbb{C} \to \mathbb{R}$**,** $f(z) = \mathrm{Re}(z) \cdot \mathrm{Im}(z)$**:** open.

$f(z) = \mathrm{Im}(\frac{1}{2}z^2)$.
$\frac{1}{2}z^2$ is open because it's holomorphic and nonconstant.
Im is open because
it maps $D_r(z)$ to the open interval $(\mathrm{Im}(z) - r, \mathrm{Im}(z) + r)$.
Hence f is a composition of open mappings,
hence open.

$f : \mathbb{C} \to \mathbb{R}$**,** $f(z) = \mathrm{Re}(z^3 + 2z)$**:** open.

$f(z) = \mathrm{Im}(iz^3 + i2z)$.
$iz^3 + i2z$ is open because it's holomorphic and nonconstant.
We showed above that Im is open.
Hence f is a composition of open mappings,
hence open.