[G] 1.

(a) Answer is 2.

$$\begin{aligned} 1 + \frac{1}{n(n+2)} &= \frac{n^2 + 2n + 1}{n(n+2)} \\ &= \frac{(n+1)^2}{n(n+2)} \end{aligned}$$

We now show by induction that

$$\prod_{n=1}^{N} \left(1 + \frac{1}{n(n+2)}\right) = \frac{2}{1}\frac{N+1}{N+2}$$

Base case: $\prod_{n=1}^{1} \left(1 + \frac{1}{n(n+2)}\right) = \frac{2}{1}$.
Induction step: for $N+1$ we have

$$\begin{aligned} \prod_{n=1}^{N+1} (\cdots) &= \frac{2}{1}\frac{N+1}{N+2}\frac{(N+2)^2}{(N+1)(N+3)} \\ &= \frac{2}{1}\frac{N+2}{N+3} \end{aligned}$$

The infinite product is now

$$\lim_{N\to\infty} \frac{2}{1}\frac{N+1}{N+2} = 2$$

(b) Answer is $\frac{1}{2}$.

$$\begin{aligned} 1 - \frac{1}{n^2} &= \frac{n^2 - 1}{n^2} \\ &= \left(\frac{n^2}{(n-1)(n+1)}\right)^{-1} \\ &= \left(1 + \frac{1}{(n-1)(n+1)}\right)^{-1} \end{aligned}$$

By part (a) we now have

$$\begin{aligned} \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) &= \prod_{n=2}^{\infty} \left(1 + \frac{1}{(n-1)(n+1)}\right)^{-1} \\ &= \prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)}\right)^{-1} \\ &= 2^{-1} \\ &= \frac{1}{2} \end{aligned}$$

(c) Answer is 4.

$$\frac{n^2-1}{n^2-4} = \frac{(n-1)(n+1)}{(n-2)(n+2)}$$

$$\begin{aligned}\prod_{n=3}^{N} \frac{n^2-1}{n^2-4} &= \prod_{n=3}^{N} \frac{(n-1)(n+1)}{(n-2)(n+2)} \\ &= \frac{4}{1}\frac{N-1}{N+2} \\ &\to 4\end{aligned}$$

The partial product formula holds by induction.
Base case is the first factor, $\frac{2\cdot 4}{1\cdot 5}$.
Induction step is $\frac{4}{1}\frac{N-1}{N+2}\frac{N(N+2)}{(N-1)(N+3)} \;= \frac{4}{1}\frac{N}{N+3}$.

[G] 10.

$$\prod_{0\le n}^{<N}\left(1+z^{2^n}\right)=\sum_{0\le k}^{<2^N} z^k$$
$$\to \frac{1}{1-z}\quad (\text{when } |z|<1)$$

The partial product formula holds by induction.
Base case is the empty product 1.
Induction step is $\left(\sum_{0\le k}^{<2^N} z^k\right)\left(1+z^{2^N}\right) = \sum_{0\le k}^{<2^N} + \sum_{k=0}^{2^N} z^{2^N} z^k = \sum_{0\le k}^{<2^{N+1}} z^k$.

[G] 14.

Throughout, $k \neq 0$.

$$\prod_{-m\le k\le tm}\left(1+\frac{z}{k}\right) = \underbrace{\prod_{-m\le k\le m}\left(1+\frac{z}{k}\right)}_{(-)}\ \underbrace{\prod_{\pm m<\pm k\le \pm tm}\left(1+\frac{z}{k}\right)^{\pm 1}}_{(--)}$$

(The sign $\pm$ in the exponent is the same sign used in the index condition $\pm m < \pm k \le \pm tm$. If $t \ge 1$ then $\pm$ is always $+$; if $t \le 1$ then $\pm$ is always $-$.)

Now

$$(-) = \prod_{k=1}^{m}\left(1-\frac{z^2}{k^2}\right) \to \frac{\sin \pi z}{\pi z}$$

and

$$\begin{aligned}\log(--) &= \pm \sum_{\pm m<\pm k\le \pm tm} \log\left(1+\tfrac{z}{k}\right)\\ &= \pm \sum_{\pm m<\pm k\le \pm tm} \tfrac{z}{k} + \underbrace{R\left(\tfrac{z}{k}\right)}_{\text{Taylor remainder}}\\ &= \pm z \underbrace{\sum_{\pm m<\pm k\le \pm tm} \tfrac{1}{k}}_{(*)} \pm \underbrace{\sum_{\pm m<\pm k\le \pm tm} R(\tfrac{z}{k})}_{(**)}\end{aligned}$$

Let M_m be the max of $\left|R(\frac{z}{k})/\frac{z}{k}\right|$ when $\pm m < \pm k \le \pm tm$
and note that $M_m \to 0$ as $m \to \infty$.
Let $r = \min\{1, t\}$.
Then

$$\begin{aligned}|(**)| &= \left|\pm \sum_{\pm m<\pm k\le \pm tm} \tfrac{z}{k}\left(R(\tfrac{z}{k})/\tfrac{z}{k}\right)\right|\\ &\le \sum_{\pm m<\pm k\le \pm tm} \tfrac{|z|}{k}\left|R(\tfrac{z}{k})/\tfrac{z}{k}\right|\\ &\le \sum_{\pm m<\pm k\le \pm tm} \tfrac{|z|}{rm} M_m\\ &\le \lceil |m-tm| \rceil \tfrac{|z|}{rm} M_m\\ &\le (|m-tm|+1)\tfrac{|z|}{rm} M_m\\ &= (|1-t| + \tfrac{1}{m})\tfrac{|z|}{r} M_m\\ &\to 0\end{aligned}$$

We also have

$$(*) = \sum_{\pm m+1\le \pm k\le \pm tm} \tfrac{1}{k}$$

by inconsequentially changing the index condition.
Using integral bounds on $(*)$, we have

$$(*) \le \pm \int_m^{tm} \tfrac{1}{x}dx = \pm \log t$$

and

$$(*) \geq \pm \int_{m}^{tm\mp 1} \frac{1}{x+1} dx = \log\left(t \mp \tfrac{1}{m}\right)$$
$$\to \pm \log t$$

so $(*) \to \pm \log t$.

Conclusion:

$$\lim_{m\to\infty} \prod_{-m\leq k\leq tm} \left(1+\tfrac{z}{k}\right) = \lim_{m\to\infty} \underbrace{\prod_{-m\leq k\leq m} \left(1+\frac{z}{k}\right)}_{(-)} \underbrace{\prod_{\pm m<\pm k\leq \pm tm} \left(1+\frac{z}{k}\right)^{\pm 1}}_{(--)}$$

$$= \frac{\sin \pi z}{\pi z} \exp\left(\lim_{m\to\infty} \pm z \underbrace{\sum_{\pm m<\pm k\leq \pm tm} \tfrac{1}{k}}_{(*)} \pm \underbrace{\sum_{\pm m<\pm k\leq \pm tm} R(\tfrac{z}{k})}_{(**)} \right)$$

$$= \frac{\sin \pi z}{\pi z} \exp\left(\pm z(\pm \log t) \pm 0\right)$$

$$= \frac{\sin \pi z}{\pi z} \exp(z \log t)$$

$$= \frac{\sin \pi z}{\pi z} t^z$$

[S] 3.

Let $\beta = \mathrm{Im}(\tau)$. **Order ≤ 2.**

$$\begin{aligned}
|\Theta(z|\tau)| &\leq \sum_{n\in\mathbb{Z}} \left| e^{\pi i n^2 \tau} e^{2\pi i n z} \right| \\
&\leq \sum_{n\in\mathbb{Z}} e^{-\pi n^2 \beta} e^{-2\pi n |z|} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{e^{-\pi\beta\left((n-\frac{|z|}{\beta})^2 - \frac{|z|^2}{\beta^2}\right)}}_{=:a_n} \\
&= \sum_{|n|<2\frac{|z|}{\beta}} a_n + \sum_{k=0}^{\infty} a_{\lceil 2\frac{|z|}{\beta}\rceil + k} + \sum_{k=0}^{\infty} a_{-(\lceil 2\frac{|z|}{\beta}\rceil + k)} \\
&\leq \sum_{|n|<2\frac{|z|}{\beta}} a_n + \frac{2}{1-e^{-2\pi|z|}}
\end{aligned}$$

The last inequality holds because

$$\left| \pm(\lceil 2\tfrac{|z|}{\beta}\rceil + k) - \tfrac{|z|}{\beta} \right| \geq \tfrac{|z|}{\beta} + k$$

$$\left(\pm(\lceil 2\tfrac{|z|}{\beta}\rceil + k) - \tfrac{|z|}{\beta}\right)^2 - \tfrac{|z|^2}{\beta^2} \;\geq 2\tfrac{|z|}{\beta}k + k^2 \;\geq 2\tfrac{|z|}{\beta}k$$

$$-\pi\beta\left(\pm(\lceil 2\tfrac{|z|}{\beta}\rceil + k) - \tfrac{|z|}{\beta}\right)^2 - \tfrac{|z|^2}{\beta^2} \;\leq -2\pi|z|k$$

$$a_{\pm(\lceil 2\frac{|z|}{\beta}\rceil + k)} = e^{-\pi\beta\left(\pm(\lceil 2\frac{|z|}{\beta}\rceil + k) - \frac{|z|}{\beta}\right)^2 - \frac{|z|^2}{\beta^2}} \;\leq e^{-2\pi|z|k}$$

$$\sum_{k=0}^{\infty} a_{\pm(\lceil 2\frac{|z|}{\beta}\rceil + k)} \;\leq \frac{1}{1-e^{-2\pi|z|}}$$

Next, observe that

$$-\pi\beta\left((n-\tfrac{|z|}{\beta})^2 - \tfrac{|z|^2}{\beta^2}\right) \;\leq -\pi\beta(-\tfrac{|z|^2}{\beta^2}) \;= \tfrac{\pi}{\beta}|z|^2$$

So for all n,

$$a_n \leq e^{\frac{\pi}{\beta}|z|^2}$$

This gives

$$\begin{aligned}
|\Theta(z|\tau)| &\leq \sum_{|n|<2\frac{|z|}{\beta}} a_n + \frac{2}{1-e^{-2\pi|z|}} \\
&\leq \sum_{|n|<2\frac{|z|}{\beta}} e^{\frac{\pi}{\beta}|z|^2} + \frac{2}{1-e^{-2\pi|z|}} \\
&\leq (2\tfrac{|z|}{\beta}+1)e^{\frac{\pi}{\beta}|z|^2} + \frac{2}{1-e^{-2\pi|z|}} \\
&\leq Ce^{|z|^2}e^{\frac{\pi}{\beta}|z|^2} + De^{(1+\frac{\pi}{\beta})|z|^2}
\end{aligned}$$

for some $C, D > 0$.
Letting $A = C + D$ and $B = 1 + \frac{\pi}{\beta}$, we now have $A, B > 0$ and

$$|\Theta(z|\tau)| \leq Ae^{B|z|^2}$$

so $z \mapsto \Theta(z|\tau)$ has order ≤ 2.

Lemma. *The function* $z \mapsto \Theta(z|\tau)$ *is not identically* 0.
Proof.
Fix z.

For each integer $N \geq 0$, define

$$\Theta_N(z|\tau) = \sum_{\substack{n \in \mathbb{Z} \\ 2^N | n}} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

where the second "|" means divisibility.
Note that $\Theta_0 = \Theta$.
Notice that $\forall z \in \mathbb{C}$:

$$\begin{aligned}
\Theta_1(z|\tau) &= \tfrac{1}{2}\left(\Theta_0(z|\tau) + \Theta_0(z + 1 \mid \tau)\right) \\
\Theta_2(z|\tau) &= \tfrac{1}{2}\left(\Theta_1(z|\tau) + \Theta_1(z + \tfrac{1}{2} \mid \tau)\right) \\
&\vdots \\
\Theta_N(z|\tau) &= \tfrac{1}{2}\left(\Theta_{N-1}(z|\tau) + \Theta_{N-1}(z + \tfrac{1}{2^{N-1}} \mid \tau)\right) \\
&\vdots
\end{aligned}$$

because $e^{i\pi} = -1$.
So if $z \mapsto \Theta(z|\tau)$ is identically 0, then so is $z \mapsto \Theta_N(z|\tau)$ for every N.
But we also have

$$\begin{aligned}
|\Theta_N(z|\tau)| &= \left| 1 + \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ 2^N | n}} e^{\pi i n^2 \tau} e^{2\pi i n z} \right| \\
&\geq 1 - \left| \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ 2^N | n}} e^{\pi i n^2 \tau} e^{2\pi i n z} \right| \\
&\geq 1 - \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ 2^N | n}} \left| e^{\pi i n^2 \tau} e^{2\pi i n z} \right| \\
&= 1 - \sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ 2^N | n}} e^{-\pi n^2 \beta} e^{2\pi n |z|} \\
&\geq 1 - \sum_{|n| \geq 2^N} e^{-\pi n^2 \beta} e^{2\pi n |z|}
\end{aligned}$$

For sufficiently large N, this gives

$$
\begin{aligned}
|\Theta_N(z|\tau)| &\geq 1 - \sum_{|n|\geq 2^N} e^{-|n|} \\
&\geq 1 - \frac{2e^{-2^N}}{1-e^{-1}} \\
&> 0
\end{aligned}
$$

So there exists N for which $z \mapsto \Theta_N(z|\tau)$ is not identically 0.
Therefore $z \mapsto \Theta(z|\tau)$ is not identically 0.

Order ≥ 2.
Completing the square on $\pi i n^2\tau + 2\pi i n z$ shows that

$$
\Theta(z|\tau) = e^{-i\tau\pi\frac{z^2}{\tau^2}} \underbrace{\sum_{n\in\mathbb{Z}} e^{i\tau\pi(n+\frac{z}{\tau})^2}}_{=:S(\frac{z}{\tau})}
$$

Notice that $\forall k \in \mathbb{Z}: \quad S\left(\frac{z}{\tau}+k\right) = S(\frac{z}{\tau})$.
Fixing $z_0 \in \mathbb{C}$ with $\Theta(z_0|\tau) \neq 0$, we have $S\left(\frac{z_0}{\tau}\right) \neq 0$ and

$$
\begin{aligned}
\Theta(z_0 + k\tau \mid \tau) &= e^{-i\tau\pi\frac{(z_0+k\tau)^2}{\tau^2}} S(\tfrac{z}{\tau}) \\
&= e^{\underbrace{\beta\pi k^2 - i\pi\left[\ \tau(2\frac{z_0}{\tau}k + \frac{z_0^2}{\tau^2}) + \alpha k^2\ \right]}_{(*)}} S(\tfrac{z}{\tau})
\end{aligned}
$$

Write $z = z_0 + k\tau$.
For sufficiently large k,

$$
\begin{aligned}
\mathrm{Re}(*) &\geq \tfrac{1}{2}\beta\pi k^2 \\
|k| &\geq \tfrac{1}{2|\tau|}|z| \\
\mathrm{Re}(*) &\geq \underbrace{\tfrac{\beta\pi}{8|\tau|^2}}_{=:B} |z|^2
\end{aligned}
$$

With $A := |S(\frac{1}{z_0}\tau)|$, we have

$$
|\Theta(z|\tau)| \geq Ae^{B|z|^2}
$$

and $A, B > 0$.
Since this occurs for any sufficiently large z of the form $z = z_0 + k$,
we conclude that $z \mapsto \Theta(z|\tau)$ has order at least 2.

Therefore $z \mapsto \Theta(z|\tau)$ has order 2.

[S] 4. (a)

A formula.

For any sequence $a_n \in \mathbb{C}$,

$$\prod_{n=1}^{N}(1+a_n) = 1+\sum_{1\le n_1\le N} a_{n_1} + \sum_{1\le n_1<n_2\le N} a_{n_1}a_{n_2} + \cdots + \sum_{1\le n_1<\cdots c_N\le N} a_{n_1}\cdots a_{n_N}$$

(This can easily be proven by induction.)
So if $a_n \ge 0$, we have the formula

$$\prod_{n=1}^{\infty}(1+a_n) = \sum_{k=0}^{\infty}\sum_{1\le n_1<\cdots c_k<\infty} a_{n_1}\cdots a_{n_k}$$

Namely, $\prod(1+a_n)$ is the sum of all products of distinctly indexed terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

Order ≤ 2.

$$\begin{aligned}|F(z)| &= \prod_{n=1}^{\infty}\left|1-e^{-2\pi nt}e^{2\pi iz}\right| \\ &\le \prod_{n=1}^{\infty}\left(1+e^{-2\pi nt}e^{2\pi|z|}\right)\end{aligned}$$

Substituting $r=e^{2\pi t}>1$ and $v=|z|/t$,
we use our formula to obtain

$$\begin{aligned}|F(z)| &\le \prod_{n=1}^{\infty}\left(1+r^{v-n}\right) \\ &= \sum_{k=0}^{\infty}\sum_{1\le n_1<\cdots c_k<\infty} r^{kv-(n_1+\cdots+n_k)}\end{aligned}$$

Claim. *Let $a\in\mathbb{Z}^+$. Then for each $k\ge 0$,*

$$\sum_{a\le n_1<\cdots c_k<\infty} r^{-(n_1+\cdots+n_k)} = r^{-k(a-1)}\prod_{j=1}^{k}(r^j-1)^{-1}$$

Proof.

Induction on k.
Base case ($k=0$): $r^{-0}=\prod_{j=1}^{0}(r^j-1)^{-1}$.

Induction step:

$$\begin{aligned}
\sum_{a\le n_1<\cdots c_{k+1}<\infty} r^{-(n_1+\cdots+n_{k+1})} &= \sum_{a\le n_1<\infty} r^{-n_1} \sum_{n_1+1\le n_2<\cdots c_{k+1}<\infty} r^{-(n_2+\cdots+n_{k+1})} \\
&= \sum_{a\le n_1<\infty} r^{-n_1} r^{-kn_1} \prod_{j=1}^{k} (r^j-1)^{-1} \\
&= r^{-(k+1)a}(1-e^{-k+1})^{-1} \prod_{j=1}^{k} (r^j-1)^{-1} \\
&= r^{-(k+1)(a-1)} \prod_{j=1}^{k+1} (r^j-1)^{-1}
\end{aligned}$$

QED.

Thus our bound is

$$\begin{aligned}
|F(z)| &\le \sum_{k=0}^{\infty} r^{kv} \sum_{1\le n_1<\cdots c_k<\infty} r^{-(n_1+\cdots+n_k)} \\
&= \sum_{k=0}^{\infty} r^{kv} r^{-k(1-1)} \prod_{j=1}^{k} (r^j-1)^{-1} \\
&= \sum_{k=0}^{\infty} r^{kv} \prod_{j=1}^{k} (r^j-1)^{-1} \\
&\le \sum_{k=0}^{\infty} r^{kv} \prod_{j=1}^{k} qr^{-j} \qquad \text{where } q=\tfrac{r}{r-1} \\
&= \sum_{k=0}^{\infty} r^{kv} q^k r^{-k(k+1)/2} \\
&= \sum_{k=0}^{\infty} \left(qr^{v-(k+1)/2}\right)^k
\end{aligned}$$

Pick n large enough that $qr^{-n}<1$.
(Note that q, n don't depend on v.)
Letting v be sufficiently large,
we have

$$\begin{aligned}
|F(z)| &\le \sum_{0\le k}^{<2v+n-1} \left(qr^{v-(k+1)/2}\right)^k + \sum_{2v+n-1\le k}^{<\infty} \left(qr^{v-(k+1)/2}\right)^k \\
&\le \sum_{0\le k}^{<2v+n-1} (qr^v)^{3v} + \sum_{2v+n-1\le k}^{<\infty} \left(qr^{-n}\right)^k \\
&\le (2v+n)q^{3v}r^{3v^2} + \frac{1}{1-qr^{-n}} \\
&\le Ar^{v^2}r^{3v^2} \\
&\le Ar^{4v^2}
\end{aligned}$$

for large enough $A > 0$.
This bound holds when v is sufficiently large;
when v isn't, it will still hold if A is chosen large enough.
Remembering our substitutions, we obtain

$$|F(z)| \leq Ae^{\frac{8\pi}{t}|z|^2}$$

QED.

Order ≥ 2.
Let

$$z = 1 + iy$$

with $y < 0$.
If y is sufficiently large, then

$$-y \geq \tfrac{1}{2}|z|$$

Hence, with $q = \frac{1}{2}|z|$,

$$\begin{aligned}
|F(z)| &= \prod_{n=1}^{\infty} \left(1 + e^{2\pi nt} e^{-2\pi y}\right) \\
&\geq \prod_{n=1}^{\infty} \left(1 + e^{2\pi nt} e^{2\pi \frac{1}{2}|z|}\right) \\
&= \prod_{n=1}^{\infty} \left(1 + r^{q-n}\right) \\
&= \sum_{k=0}^{\infty} r^{kq} \prod_{j=1}^{k} (r^j - 1)^{-1} \\
&\geq \sum_{k=0}^{\infty} r^{kq} \prod_{j=1}^{k} (r^j)^{-1} \\
&\geq \sum_{k=0}^{\infty} r^{kq - k(k+1)/2} \\
&\geq \sum_{k=0}^{\infty} \left(r^{q-(k+1)/2}\right)^k
\end{aligned}$$

For q sufficiently large, there exists an integer $K \geq 0$ with $\frac{1}{3}q \leq K \leq \frac{2}{3}q - 1$, hence

$$\begin{aligned}
|F(z)| &\geq \left(r^{q-(K+1)/2}\right)^K \\
&\geq \left(r^{q-\frac{1}{3}q}\right)^K \\
&= \left(r^{\frac{2}{3}q}\right)^K \\
&\geq \left(r^{\frac{2}{3}q}\right)^{\frac{1}{3}q} \\
&= r^{\frac{2}{9}q^2} \\
&= e^{\frac{\pi}{9t}|z|^2}
\end{aligned}$$

Since this lower bound holds whenever $\mathrm{Re}(z) = 1$, $\mathrm{Im}(z) < 0$, and y sufficiently large,
it holds for arbitrarily large z,
so F is of order ≥ 2.

Hence F has order 2.