[G] p. 356 ex 15.

For $n \geq 0$ define

$$a_n(z) = \begin{cases} 0 & n = 0 \\ \frac{n}{z+n}\left(\frac{n+1}{n}\right)^z & n > 0 \end{cases}$$

Lemma. *Let $N \in \mathbb{Z}^+$. The product*

$$\prod_{N \leq n}^{<\infty} a_n(z)$$

converges on $D_N(0)$ to a holomorphic nonvanishing function.

Proof.

The factors a_n are holomorphic on $D_N(0)$,
hence it suffices to show uniform nonzero convergence of the product on compact subsets of $D_N(0)$.
So let $C \subseteq D_N(0)$ be compact, and let $R < N$ with $|z| \leq R$ on C.

For each n, z:

$$\begin{aligned} a_n(z) &= \frac{n}{z+n}\left(\frac{n+1}{n}\right)^z \\ &= \left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^z \\ &= e^{-\sum_{1\leq k}^{<\infty}(-1)^{k+1}\frac{z^k}{n^k}/k} \quad e^{\sum_{1\leq k}^{<\infty}(-1)^{k+1}\frac{z}{n^k}/k} \\ &= e^{\sum_{2\leq k}^{<\infty}(-1)^{k+1}\frac{z-z^k}{n^k}/k} \\ &=: e^{b_n(z)} \end{aligned}$$

So the product is

$$\prod_{N\leq n}^{<\infty} e^{b_n(z)} = e^{\sum_{N\leq n}^{<\infty} b_n(z)}$$

Defining

$$c_n = \left(\frac{R}{1-1/N} + \frac{R^2}{1-R/N}\right)\frac{1}{n^2}$$

we find that $c_n \geq 0$, $\sum c_n < \infty$, and

$$\forall n \geq N \ \forall z \in C\colon \ |b_n(z)| \leq c_n$$

We justify this inequality.

$$\begin{aligned}
|b_n(z)| &\leq \sum_{2\leq k}^{<\infty} \frac{|z-z^k|}{n^k}/k \\
&\leq \sum_{2\leq k}^{<\infty} \frac{|z|}{n^k}/k + \sum_{2\leq k}^{<\infty} \frac{|z|^k}{n^k}/k \\
&\leq \frac{|z|/n^2}{1-1/n} + \frac{|z|^2/n^2}{1-|z|/n} \\
&\leq \frac{R/n^2}{1-1/N} + \frac{R^2/n^2}{1-R/N} \\
&= \left(\frac{R}{1-1/N} + \frac{R^2}{1-R/N}\right)\frac{1}{n^2} \\
&= c_n
\end{aligned}$$

Write $S = \sum_{N\leq n}^{<\infty} c_n$.
For any integers $P, Q \geq N$ (WLOG $P \leq Q$) and any $z \in C$:

$$\begin{aligned}
&\left|\prod_{N\leq n}^{<P} a_n(z) - \prod_{N\leq n}^{<Q} a_n(z)\right| \\
&= \left|\prod_{N\leq n}^{<P} a_n(z)\right| \left|1 - \prod_{P\leq n}^{<Q} a_n(z)\right| \\
&\leq \prod_{N\leq n}^{<P} e^{|b_n(z)|} \left|1 - \prod_{P\leq n}^{<Q} e^{b_n(z)}\right| \\
&\leq e^{\sum_{N\leq n}^{<\infty} c_n} \left|1 - e^{\sum_{P\leq n}^{<Q} b_n(z)}\right|
\end{aligned}$$

Since $\sum b_n(z)$ converges uniformly,
we find that as $P \to \infty$,
$e^{\sum_{P\leq n}^{<Q} b_n(z)} \to 1$ uniformly,
hence the above expression converges uniformly to 0.

This shows that the sequence of partial products is uniformly Cauchy,
hence uniformly converges.

It only remains to show that the product is nonvanishing.

This comes by

$$\begin{aligned}\left|\prod_{N\leq n}^{<\infty} a_n(z)\right| &= \prod_{N\leq n}^{<\infty} |e^{b_n(z)}| \\ &\geq \prod_{N\leq n}^{<\infty} e^{-|b_n(z)|} \\ &\geq \prod_{N\leq n}^{<\infty} e^{-|c_n|} \\ &= e^{\sum_{N\leq n}^{<\infty} -|c_n|} \\ &> 0\end{aligned}$$

since the infinite sum is finite.

QED.

Convergence of Γ to a meromorphic function whose poles are simple poles at each point of $\mathbb{Z}_{\leq 0}$.

For each $k \in \mathbb{Z}_{\geq 0}$, define on $D_{k+1}(0)$ a function

$$\Gamma_k = a_0 \cdots a_{k-1} \prod_{k+1\leq n}^{<\infty} a_n$$

This is a product of finitely many meromorphic functions ($k+1$ of them),
each nonvanishing and finite at $-k$,
and each finite outside of $\mathbb{Z}_{\leq 0}$.

Furthermore, a_k is a meromorphic function with a simple pole at $-k$,
and no other poles.
Hence the function

$$\Gamma = a_k \cdot \left(a_0 \cdots a_{k-1} \prod_{k+1\leq n}^{<\infty} a_n \right)$$

is meromorphic on $D_{k+1}(0)$ with a simple pole at $-k$,
and no poles outside $\mathbb{Z}_{\leq 0}$.

Since k was arbitrary,
we find that Γ is meromorphic on $\mathbb{C} = \bigcup D_{k+1}(0)$
and its poles are simple poles at each point of $\mathbb{Z}_{\leq 0}$.

Limit.

The m-th partial product is

$$\begin{aligned}\prod_{0\leq n}^{<n} a_n(z) &= \frac{1}{z}\frac{1}{z+1}\frac{2}{z+2}\cdots\frac{m-1}{z+m-1}\left(\frac{2}{1}\frac{3}{2}\cdots\frac{m}{m-1}\right)^z \\ &= \frac{(m-1)!m^z}{z(z+1)\cdots(z+m-1)}\end{aligned}$$

Therefore

$$\Gamma(z) = \lim_{m\to\infty} \frac{(m-1)!m^z}{z(z+1)\cdots(z+m-1)}$$

Recursion relation.

$$\begin{aligned}
\Gamma(z+1) &= \lim_{m\to\infty} \frac{(m-1)!m^{(z+1)}}{(z+1)(z+2)\cdots(z+m)} \\
&= \lim_{m\to\infty} \frac{m!m^z}{(z+1)(z+2)\cdots(z+m)} \\
&= z \lim_{m\to\infty} \frac{m!m^z}{z(z+1)(z+2)\cdots(z+m)} \\
&= z \lim_{m\to\infty} \frac{(m-1)!(m-1)^z}{z(z+1)(z+2)\cdots(z+m-1)} \\
&= z\Gamma(z)
\end{aligned}$$

Factorial.

Holds for $n = 0$, since

$$\begin{aligned}
\Gamma(0+1) &= \lim_{m\to\infty} \frac{(m-1)!m^1}{1\cdot 2\cdots m} \\
&= \lim_{m\to\infty} 1 \\
&= 1 \\
&= 0!
\end{aligned}$$

For each subsequent n, we have

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \\
&= n\ (n-1)! \\
&= n!
\end{aligned}$$

QED.

[G] p. 356 ex 16.

Convention.

By default, sums and products range only over those k for which $\alpha_k \neq 0$.
However, other statements and expressions involving k will, by default, range over all k.

(a)

Preliminaries.

Let $m = \#\{k \mid \alpha_k = 0\}$.
(This is finite because $|\alpha_k| \to 1$ because $\sum_{\text{all } k}(1 - |\alpha_k|) < \infty$.)

The product in question will be written as

$$B(z) = z^m \prod \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z}$$

(We will show that the product converges in $\mathbb{C}^*$ on $\mathbb{C}^* \setminus E$, hence that B is defined there.)

Regard each factor of B as a meromorphic function on $\mathbb{C}^*$.

Lemma. $\prod |\alpha_k|$ *converges in* $\mathbb{C} \setminus \{0\}$.

Proof of lemma.

$|\alpha_k| = 1 - (1 - |\alpha_k|)$ and we're given that $\sum(1 - |\alpha_k|)$ converges (absolutely, since its terms are nonnegative), hence the product converges in $\mathbb{C}$ and is 0 iff one of its factors is.
None of its factors is 0 (since the product is taken over all k with $\alpha_k \neq 0$) so the product is nonzero, completing the proof of the lemma.

More preliminaries.

Observe that

$$\begin{aligned}\overline{\alpha_k}^{-1} - \alpha_k &= \overline{\alpha_k}^{-1}(1 - |\alpha_k|^2) \\ &= \overline{\alpha_k}^{-1}(1 + |\alpha_k|)(1 - |\alpha_k|)\end{aligned}$$

We calculate in parallel.

$$\begin{aligned}&\frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z} \\ &= \frac{\alpha_k - z}{\overline{\alpha_k}^{-1} - z} |\alpha_k|^{-1} \\ &= \left(1 + \frac{\alpha_k - \overline{\alpha_k}^{-1}}{\overline{\alpha_k}^{-1} - z}\right) |\alpha_k|^{-1} \\ &= \left(1 - \frac{\overline{\alpha_k}^{-1}(1 + |\alpha_k|)}{\overline{\alpha_k}^{-1} - z}(1 - |\alpha_k|)\right) |\alpha_k|^{-1}\end{aligned}$$

$$\begin{aligned}&\left(\frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z}\right)^{-1} \\ &= \frac{\overline{\alpha_k}^{-1} - z}{\alpha_k - z} |\alpha_k| \\ &= \left(1 + \frac{\overline{\alpha_k}^{-1} - \alpha_k}{\alpha_k - z}\right) |\alpha_k| \\ &= \left(1 + \frac{\overline{\alpha_k}^{-1}(1 + |\alpha_k|)}{\alpha_k - z}(1 - |\alpha_k|)\right) |\alpha_k|\end{aligned}$$

Hence, writing $c = \prod |\alpha_k|$,

$$B(z) = c^{-1}(1-(1-z))^m \quad \prod \left(1 - \underbrace{\frac{\overline{\alpha_k}^{-1}(1+|\alpha_k|)}{\overline{\alpha_k}^{-1} - z}(1-|\alpha_k|)}_{=:f_k(z)} \right)$$

$$\frac{1}{B}(z) = c \quad (1-(1-z^{-1}))^m \prod \left(1 + \underbrace{\frac{\overline{\alpha_k}^{-1}(1+|\alpha_k|)}{\alpha_k - z}(1-|\alpha_k|)}_{=:g_k(z)} \right)$$

Here we have defined f_k and g_k for each k with $\alpha_k \neq 0$.
For k with $\alpha_k = 0$, define $f_k(z) = 1 - z$ and $g_k(z) = (1 - z^{-1})$.
This allows us to write

$$B(z) = c^{-1} \prod_{\text{all } k} (1 - f_k(z))$$
$$\frac{1}{B}(z) = c \quad \prod_{\text{all } k} (1 - g_k(z))$$

Now define $Z = \{\alpha_k\}$ and $P = \left\{\overline{\alpha_k}^{-1}\right\}$, allowing $\alpha_k = 0$.
These are the zeros and poles, respectively, of the factors of B, as well as of f_k.
Hence P are the poles of the f_k
and Z are the poles of the g_k.

And finally, observe that since $|\alpha_k| \to 1$,
the accumulation points of α_k on $\partial\mathbb{D}$
are precisely the accumulation points of α_k, which
are precisely the accumulation points of $\overline{\alpha_k}^{-1}$

Theorem. *B converges normally on $\mathbb{C}^* \setminus E$ as $K \to \infty$.*

Gamelin defines the phrase "converges normally" on p. 316;
on the same page are some relevant facts about the spherical metric σ.
(We will only use the Euclidean metric, which will suffice
due to the inversion and equivalence facts on that page.)

Proof of theorem.
Let $C \subseteq \mathbb{C}^* \setminus E$ be compact.

Case 1: $C \cap P = \emptyset$.
There exist constants $A, R > 0$ such that for all $z \in C$ and all sufficiently large k:

$$|\overline{\alpha_k}^{-1} - z| \geq A$$
$$|\alpha_k|^{-1} \leq R$$

For k such that $\alpha_k \neq 0$, these inequalities imply that

$$|f_k(z)| \leq \frac{2R}{A}(1 - |\alpha_k|)$$

(Here we've used the fact that $\alpha_k \in \mathbb{D}$.)
Now let

$$u_k = \begin{cases} \frac{2R}{A}(1-|\alpha_k|) & |f_k(z)| \le \frac{2R}{A}(1-|\alpha_k|) \\ \max_C |f_k| & \text{otherwise} \end{cases}$$

the second case occurs for finitely many k;
the maximum exists because C is compact and has no poles of the f_k.

Observe that $u_k \ge 0$, $\sum u_k < \infty$, and $\forall k \forall z \in C\colon\ |f_k(z)| \le u_k$.
Hence the product

$$\prod_{\text{all } k} (1 - f_k(z))$$

converges uniformly on C (Euclidean metric),
and so the product

$$B(z) = c^{-1} \prod_{\text{all } k} (1 - f_k(z))$$

converges uniformly on C (Euclidean metric), and hence is holomorphic.
(**Note for later** that B is 0 iff only one of its factors is.)

Since B is holomorphic on C, we see that $B(C)$ is a compact subset of $\mathbb{C}$,
hence $\sigma|_{B(C)\times B(C)}$ is equivalent to the Euclidean metric,
hence B converges uniformly on C with respect to σ, as desired.

Case 2: $C \cap Z = \emptyset$.
This will be similar to the previous case.

There exist constants $A, R > 0$ such that for all $z \in C$ and all sufficiently large k:

$$|\alpha_k - z| \ge A$$
$$|\alpha_k|^{-1} \le R$$

For k such that $\alpha_k \neq 0$, these inequalities imply that

$$|g_k(z)| \le \frac{2R}{A}(1 - |\alpha_k|)$$

(Here we've used the fact that $\alpha_k \in \mathbb{D}$.)
Now let

$$v_k = \begin{cases} \frac{2R}{A}(1-|\alpha_k|) & |g_k(z)| \le \frac{2R}{A}(1-|\alpha_k|) \\ \max_C |v_k| & \text{otherwise} \end{cases}$$

the second case occurs for finitely many k;
the maximum exists because C is compact and has no poles of the g_k.

Observe that $v_k \ge 0$, $\sum v_k < \infty$, and $\forall k \forall z \in C\colon\ |g_k(z)| \le v_k$.
Hence the product

$$\prod_{\text{all } k} (1 - g_k(z))$$

converges uniformly on C (Euclidean metric),
and so the product

$$\frac{1}{B}(z) = c \prod_{\text{all } k} (1 - g_k(z))$$

converges uniformly on C (Euclidean metric), and hence is holomorphic.
(**Note for later** that $\frac{1}{B}$ is 0 iff only one of its factors is,
hence B is ∞ iff one of its factors is.)

Since $\frac{1}{B}$ is holomorphic on C, we see that $\frac{1}{B}(C)$ is a compact subset of $\mathbb{C}$,
hence $\sigma|_{\frac{1}{B}(C)\times\frac{1}{B}(C)}$ is equivalent to the Euclidean metric.
Noting also the identity

$$\forall z, w \in \mathbb{C}^*: \ \sigma(z, w) = \sigma(\tfrac{1}{z}, \tfrac{1}{w})$$

we find that B converges uniformly on C with respect to σ, as desired.

Case 3: otherwise.
$C = C_1 \cup C_2$
for some compact subsets C_1, C_2 disjoint from P, Z respectively,
hence B converges uniformly on C by the previous cases.

Since C was arbitrary,
we find that the product converges uniformly on every compact subset of $\mathbb{C}^* \setminus E$,
proving the theorem.

Conclusion.

By a theorem of Gamelin, we now see that B is meromorphic or identically ∞.
Hence it is meromorphic (since it's bounded on $\mathbb{D}$, as we will see presently).

Examining the formula

$$B(z) = \prod_{\text{all } k} \begin{cases} \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z} & \alpha_k \neq 0 \\ z & \alpha_k = 0 \end{cases}$$

and recalling SS ch 1 ex 7 from hw 1,
we find that each factor of B has magnitude
< 1 if $z \in \mathbb{D}$
$= 1$ if $z \in \partial\mathbb{D}$
hence $|B|$ is < 1 on $\mathbb{D}$ and $= 1$ on $\partial\mathbb{D}$.

Finally, recalling **Note for later** from Cases 1 and 2 of our theorem,
we find that the zeros of B are precisely α_k
and the poles are precisely $\overline{\alpha_k}^{-1}$.

(b)

Let $C \subseteq \mathbb{D}$ be compact.
Let $R < 1$ with $|z| \leq R$ on C.

Magnitude of Blaschke factor. *For all $z \in C, w \in \mathbb{D}$:*

$$\left| \frac{w - z}{1 - \overline{w} z} \right| \leq |w|^{1/2}$$

Proof.

Let $z \in C, w \in \mathbb{D}$.

Write $z = x + iy, w = u + iv$.

$$\begin{aligned}
0 &\le \left[|w|^2 - 1\right]\left[|z|^2 - 1 + (1 - R^2)\right] \\
0 &\le 1 + |w|^2|z|^2 - |w|^2 - |z|^2 - (1 - R^2)(1 - |w|^2) \\
|w|^2 + |z|^2 &\le 1 + |w|^2|z|^2 - (1 - R^2)(1 - |w|^2) \\
|w|^2 + |z|^2 - 2ux - 2vy &\le 1 + |w|^2|z|^2 - 2(ux + vy) - (1 - R^2)(1 - |w|^2) \\
|w - z|^2 &\le |1 - \overline{w}z|^2 - (1 - R^2)(1 - |w|^2) \\
\left|\frac{w - z}{1 - \overline{w}z}\right|^2 &\le 1 - \frac{(1 - R^2)(1 + |w|)}{|1 - \overline{w}z|^2}(1 - |w|) \\
&\le 1 - \frac{(1 - R^2)}{(1 + R)^2}(1 - |w|) \\
&\le 1 - (1 - |w|) \\
&\le |w| \\
\left|\frac{w - z}{1 - \overline{w}z}\right| &\le |w|^{1/2}
\end{aligned}$$

QED.

Lemma. *Let $a_n \ge 0$ and $\sum_{n<\infty} a_n = +\infty$. Then $\prod_{n<\infty}(1 + a_n) = +\infty$.*
Proof.

Expanding partial products,

$$\begin{aligned}
\prod_{n<N}(1 + a_n) &= 1 + \sum_{n<N} a_n + \text{(other nonnegative terms)} \\
&\ge \sum_{n<N} a_n
\end{aligned}$$

hence $\prod_{n<\infty}(1 + a_n) \ge \sum_{n<\infty} a_n = +\infty$, QED.

Lemma. *Suppose there are finitely many k such that $\alpha_k = 0$. Then*

$$\prod |\alpha_k| = 0$$

where the product only uses those k for which $\alpha_k \ne 0$.
Proof.

Combining the hypothesis with the fact that

$$\sum_{\text{all } k}(1 - |\alpha_k|) = +\infty$$

we find that

$$\sum (1 - |\alpha_k|) = +\infty$$

hence

$$\begin{aligned}
\prod |\alpha_k|^{-1} &= \prod (1 + |\alpha_k|^{-1}(1 - |\alpha_k|)) \\
&\le \prod (1 + (1 - |\alpha_k|)) \\
&= +\infty
\end{aligned}$$

$$\begin{aligned}\prod |\alpha_k| &= \left(\prod |\alpha_k|^{-1}\right)^{-1}\\ &= (+\infty)^{-1}\\ &= 0\end{aligned}$$

QED.

Uniform convergence to 0.

The K-th partial product satisfies

$$\left| \prod_{\text{all } k<K} \begin{cases} \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1-\overline{\alpha_k} z} & \alpha_k \neq 0 \\ z & \alpha_k = 0 \end{cases} \right| \leq \prod_{\text{all } k<K} \begin{cases} |\alpha_k|^{1/2} & \alpha_k \neq 0 \\ R & \alpha_k = 0 \end{cases}$$

and the RHS approaches 0 as $K \to \infty$. Reason:

If there are infinitely many k such that $\alpha_k = 0$,
then RHS $\to 0$ because $R < 1$ and each α_k satisfies $|\alpha_k|^{1/2} \leq 1$.
Otherwise, RHS $\to 0$ because $R \leq 1$ and $\prod |\alpha_k| = 0$.

Since the magnitude of the product of interest is bounded above by RHS *for all* z,
the product converges uniformly to 0.

Since C was an arbitary compact subset of $\mathbb{D}$,
the products converge uniformly to 0 on every compact subset of $\mathbb{D}$.
QED.

[G] p. 360 2.

Construction:

$$f(z) = \prod_{0 \leq n}^{<\infty} a_n(z)$$

where

$$a_n(z) = \begin{cases} z & n = 0 \\ 1 - \frac{z}{n^2} & n > 0 \end{cases}$$

Note that each a_n is entire.

Convergence and zeros:

Let $C \subseteq \mathbb{C}$ be compact.
Let $R > 0$ with $|z| \leq R$ on C.
Defining

$$c_n = \begin{cases} B + 1 & n = 0 \\ B/n^2 & n > 0 \end{cases}$$

we find that $c_n \geq 0$, $\sum_{0 \leq n}^{<\infty} c_n < \infty$, and

$$|a_n(z) - 1| \leq c_n$$

for any $z \in C$.

By a theorem of SS,
this shows that the product converges uniformly to a holomorphic function on C
and is 0 precisely when one of its factors is.

Since $C \subseteq \mathbb{C}$ is arbitrary,
this shows that the product converges to an entire function
whose zeros are precisely n^2 for $n \geq 0$.

To see why the zeros are simple,
let $n_0 \geq 0$ and consider the function

$$f_{n_0}(z) = \prod_{\substack{0 \leq n < \infty \\ n \neq n_0}} a_n(z)$$

By essentially the same argument as above,
we find that this function is entire and does not vanish at n_0.
Observing that $f = a_{n_0} f_{n_0}$,
we find that f has exactly as many zeros at n_0 as a_{n_0} does,
hence it has exactly one, by the definition of a_{n_0}.

[G] p. 360 5.

Construction.

$$f(z) = z \prod_{r\in(0,\infty)} \prod_{\substack{u\in\mathbb{Z}[i]\\|u|=r\\\arg u\in[0,\pi/2)}} \left(1 - \frac{z^4}{u^4}\right)$$

Entirety and zeros.

Let $C \subseteq \mathbb{C}$ be compact, and let $R > 0$ with $|z| \le R$ on C.
For any $z \in C$ and any u in $\mathbb{Z}[i] \setminus \{0\}$, we have

$$|z - 1| \le R + 1$$

$$\left|-\frac{z^4}{u^4}\right| \le \frac{B^4}{|u|^4}$$

Hence, to prove that the product converges uniformly
to a holomorphic function
which vanishes precisely when one of its factors does,
it suffices to show that

$$(R{+}1) + \sum_{r\in(0,\infty)} \sum_{\substack{u\in\mathbb{Z}[i]\\|u|=r\\\arg u\in[0,\pi/2)}} \left(1 - \frac{R^4}{|u|^4}\right) < \infty$$

Proof of the above convergence:

$$\begin{aligned}
&(R+1) + \sum_{r\in[1,\infty)} \sum_{\substack{u\in\mathbb{Z}[i]\\|u|=r\\\arg u\in[0,\pi/2)}} \frac{R^4}{|u|^4} \\
&\le (R+1) + R^4 \sum_{n=1}^{\infty} \sum_{r\in[n,n+1)} \sum_{\substack{u\in\mathbb{Z}[i]\\|u|=r\\\arg u\in[0,\pi/2)}} \frac{1}{n^4} \\
&= (R+1) + R^4 \sum_{n=1}^{\infty} \frac{\#\ \mathbb{Z}[i] \cap (D_{n+1}(0) \setminus D_n(0))}{n^4} \\
&\le (R+1) + R^4 \sum_{n=1}^{\infty} \frac{4n^2 + 8n + 4}{n^4} \\
&< \infty
\end{aligned}$$

The second-to-last inequality comes from covering an annulus with a square:

$$D_{n+1}(0) \setminus D_n(0) \quad \subseteq \quad (-(n{+}1), (n{+}1))^2$$

$$\begin{aligned}
&\#\quad \mathbb{Z}[i] \ \cap\ D_{n+1}(0) \setminus D_n(0) \\
\le\ &\#\quad \mathbb{Z}[i] \ \cap\ (-(n{+}1),\ (n{+}1))^2 \\
\le\ &[(n{+}1) - -(n{+}1)]^2 \\
=\ &4n^2 + 8n + 4
\end{aligned}$$

We have now proven that the product
converges uniformly on C
to a holomorphic function f
that vanishes precisely when one of its factors does.
Since $C \subseteq \mathbb{C}$ was an arbitrary compact set,
f is entire.

$f^{-1}(0) = \mathbb{Z}[i]$ and all zeros are simple.
We factor the factors of f.

z is linear; its only root is 0, simple.
All other factors are in one-to-one correspondence
with the members of the quadrant

$$S = \{u \in \mathbb{Z}[i] \setminus \{0\} \mid 0 \leq \arg u < \pi/2\}$$

and we have the factorization

$$1 - \frac{z^4}{u^4} = \frac{1}{u^4}(u - z)(iu - z)(i^2u - z)(i^3u - z)$$

hence its roots are simple and they are $u, \ldots i^3u$,
and no other factor has roots at those points.

This proves that f has a root at every Gaussian integer and nowhere else.
To show that each root is simple,
we let $u \in \mathbb{Z}[i]$ and write

$$a_u(z)$$

to denote the factor in the product formula for f
that has a root at u.
For example, $a_0(z) = z$ and $a_{2i}(z) = (1 - \frac{z^4}{16})$.
If, in the product formula for f,
we replace a_u with $a_u/(u - z)$ (with the singularity removed),
we can show by essentially the same argument as before
that the new function f_u defined by this product
vanishes only when one of its factors does,
hence does not vanish at u,
and therefore our function $f = (u - z)f_u$ has a simple root at u.