Now we recall the simple fact that whenever $n \neq 0$, the integral of $e^{-in\theta}$ over any circle centered at the origin vanishes. Therefore

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} [u(re^{i\theta}) - Cr^s] e^{-in\theta}\, d\theta \qquad \text{when } n > 0,$$

hence

$$|a_n| \leq \frac{1}{\pi r^n} \int_0^{2\pi} [Cr^s - u(re^{i\theta})]\, d\theta \leq 2Cr^{s-n} - 2\text{Re}(a_0) r^{-n}.$$

Letting r tend to infinity along the sequence given in the hypothesis of the lemma proves that $a_n = 0$ for $n > s$. This completes the proof of the lemma and of Hadamard's theorem.

6 Exercises

1. Give another proof of Jensen's formula in the unit disc using the functions (called Blaschke factors)

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

[Hint: The function $f/(\psi_{z_1} \cdots \psi_{z_N})$ is nowhere vanishing.]

2. Find the order of growth of the following entire functions:

(a) $p(z)$ where p is a polynomial.

(b) e^{bz^n} for $b \neq 0$.

(c) e^{e^z}.

3. Show that if τ is fixed with $\text{Im}(\tau) > 0$, then the Jacobi theta function

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

is of order 2 as a function of z. Further properties of Θ will be studied in Chapter 10.

[Hint: $-n^2 t + 2n|z| \leq -n^2 t/2$ when $t > 0$ and $n \geq 4|z|/t$.]

4. Let $t > 0$ be given and fixed, and define $F(z)$ by

$$F(z) = \prod_{n=1}^{\infty} (1 - e^{-2\pi n t} e^{2\pi i z}).$$

Note that the product defines an entire function of z.

(a) Show that $|F(z)| \leq Ae^{a|z|^2}$, hence F is of order 2.

(b) F vanishes exactly when $z = -int + m$ for $n \geq 1$ and n, m integers. Thus, if z_n is an enumeration of these zeros we have

$$\sum \frac{1}{|z_n|^2} = \infty \quad \text{but} \quad \sum \frac{1}{|z_n|^{2+\epsilon}} < \infty.$$

[Hint: To prove (a), write $F(z) = F_1(z)F_2(z)$ where

$$F_1(z) = \prod_{n=1}^{N}(1 - e^{-2\pi nt}e^{2\pi iz}) \quad \text{and} \quad F_2(z) = \prod_{n=N+1}^{\infty}(1 - e^{-2\pi nt}e^{2\pi iz}).$$

Choose $N \approx c|z|$ with c appropriately large. Then, since

$$\left(\sum_{N+1}^{\infty} e^{-2\pi nt}\right) e^{2\pi|z|} \leq 1\,,$$

one has $|F_2(z)| \leq A$. However,

$$|1 - e^{-2\pi nt}e^{2\pi iz}| \leq 1 + e^{2\pi|z|} \leq 2e^{2\pi|z|}.$$

Thus $|F_1(z)| \leq 2^N e^{2\pi N|z|} \leq e^{c'|z|^2}$. Note that a simple variant of the function F arises as a factor in the triple product formula for the Jacobi theta function Θ, taken up in Chapter 10.]

5. Show that if $\alpha > 1$, then

$$F_\alpha(z) = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi izt}\, dt$$

is an entire function of growth order $\alpha/(\alpha - 1)$.
[Hint: Show that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq c|z|^{\alpha/(\alpha-1)}$$

by considering the two cases $|t|^{\alpha-1} \leq A|z|$ and $|t|^{\alpha-1} \geq A|z|$, for an appropriate constant A.]

6. Prove Wallis's product formula

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)\cdot(2m+1)}\cdots.$$

[Hint: Use the product formula for $\sin z$ at $z = \pi/2$.]

7. Establish the following properties of infinite products.

(a) Show that if $\sum |a_n|^2$ converges, then the product $\prod(1+a_n)$ converges to a non-zero limit if and only if $\sum a_n$ converges.

(b) Find an example of a sequence of complex numbers $\{a_n\}$ such that $\sum a_n$ converges but $\prod(1+a_n)$ diverges.

(c) Also find an example such that $\prod(1+a_n)$ converges and $\sum a_n$ diverges.

8. Prove that for every z the product below converges, and

$$\cos(z/2)\cos(z/4)\cos(z/8)\cdots = \prod_{k=1}^{\infty}\cos(z/2^k) = \frac{\sin z}{z}.$$

[Hint: Use the fact that $\sin 2z = 2\sin z\cos z$.]

9. Prove that if $|z| < 1$, then

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \prod_{k=0}^{\infty}(1+z^{2^k}) = \frac{1}{1-z}.$$

10. Find the Hadamard products for:

(a) $e^z - 1$;

(b) $\cos \pi z$.

[Hint: The answers are $e^{z/2} z \prod_{n=1}^{\infty}(1 + z^2/4n^2\pi^2)$ and $\prod_{n=0}^{\infty}(1 - 4z^2/(2n+1)^2)$, respectively.]

11. Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem.

[Hint: If f misses a, then $f(z) - a$ is of the form $e^{p(z)}$ where p is a polynomial.]

12. Suppose f is entire and never vanishes, and that none of the higher derivatives of f ever vanish. Prove that if f is also of finite order, then $f(z) = e^{az+b}$ for some constants a and b.

13. Show that the equation $e^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.

[Hint: Apply Hadamard's theorem.]

14. Deduce from Hadamard's theorem that if F is entire and of growth order ρ that is non-integral, then F has infinitely many zeros.

15. Prove that every meromorphic function in $\mathbb{C}$ is the quotient of two entire functions. Also, if $\{a_n\}$ and $\{b_n\}$ are two disjoint sequences having no finite limit points, then there exists a meromorphic function in the whole complex plane that vanishes exactly at $\{a_n\}$ and has poles exactly at $\{b_n\}$.

16. Suppose that

$$Q_n(z) = \sum_{k=1}^{N_n} c_k^n \, z^k$$

are given polynomials for $n = 1, 2, \ldots$. Suppose also that we are given a sequence of complex numbers $\{a_n\}$ without limit points. Prove that there exists a meromorphic function $f(z)$ whose only poles are at $\{a_n\}$, and so that for each n, the difference

$$f(z) - Q_n\left(\frac{1}{z - a_n}\right)$$

is holomorphic near a_n. In other words, f has a prescribed poles and principal parts at each of these poles. This result is due to Mittag-Leffler.

17. Given two countably infinite sequences of complex numbers $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$, with $\lim_{k\to\infty} |a_k| = \infty$, it is always possible to find an entire function F that satisfies $F(a_k) = b_k$ for all k.

(a) Given n distinct complex numbers $a_1, \ldots, a_n$, and another n complex numbers $b_1, \ldots, b_n$, construct a polynomial P of degree $\leq n - 1$ with

$$P(a_i) = b_i \qquad \text{for } i = 1, \ldots, n.$$

(b) Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of distinct complex numbers such that $a_0 = 0$ and $\lim_{k\to\infty} |a_k| = \infty$, and let $E(z)$ denote a Weierstrass product associated with $\{a_k\}$. Given complex numbers $\{b_k\}_{k=0}^{\infty}$, show that there exist integers $m_k \geq 1$ such that the series

$$F(z) = \frac{b_0}{E'(z)} \frac{E(z)}{z} + \sum_{k=1}^{\infty} \frac{b_k}{E'(a_k)} \frac{E(z)}{z - a_k} \left(\frac{z}{a_k}\right)^{m_k}$$

defines an entire function that satisfies

$$F(a_k) = b_k \qquad \text{for all } k \geq 0.$$

This is known as the Pringsheim interpolation formula.

7 Problems

1. Prove that if f is holomorphic in the unit disc, bounded and not identically zero, and $z_1, z_2, \ldots, z_n, \ldots$ are its zeros ($|z_k| < 1$), then

$$\sum_n (1 - |z_n|) < \infty.$$