# MATH 214 Homework 1

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## 1 Problem 1-4

### (a)

Because $\mathcal{U} = \{U_\alpha\}$ is an open cover of the topological manifold $M$, for each point $p \in M$, there exists an open set $U_\alpha \in \mathcal{U}$ such that $p \in U_\alpha$. Because $U_\alpha$ intersects only finitely many other sets in the coveer, it is evident that $\mathcal{U}$ is locally finite by definition.

### (b)

Let $M = (0,2) \times \mathbb{R}$ and $\mathcal{U} = \{(1-\epsilon, 2) \times \mathbb{R}\} \cup \{(0,1) \times (n-\epsilon, n+1+\epsilon) : n \in \mathbb{Z}\}$ for some $\epsilon \in (0,1)$. It is obvious that each point on $M$ has a neighborhood, which can be chosen to be small enough, that intersects only finitely many sets in the cover. However, $(1-\epsilon, 2) \times \mathbb{R}$ intersects all other sets.

### (c)

Because $\mathcal{U}$ is locally finite, for each point $p \in M$, there exists an open neighborhood $V_p$ which intersects finitely many sets in the cover $\mathcal{U}$. Those sets $V_p$'s form another cover on $M$. By the precompactness of $U_\alpha \in \mathcal{U}$, $\bar{U}_\alpha$ can be covered by finitely many $V_p$'s, say, $U_\alpha \subset \bar{U}_\alpha \subset \bigcup_{i=1}^n V_{p_i}$. Because each $V_{p_i}$ intersects only finitely many $U_\beta$'s, it means $U_\alpha$ only intersects finitely many other sets in the cover $\mathcal{U}$.

## 2 Problem 1-6

First consider the map $F_s(x) = |x|^{s-1}x$ for all $s > 0$. $\forall x \in \mathbb{B}^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$, $|F_s(x)| = |x|^s \leq 1$, then $F_s : \mathbb{B}^n \to \mathbb{B}^n$. Because $x, F_s(x)$ only differ by a scaling factor $|x|^{s-1} \in (0,1]$, it is evident that $F_s$ is injective and continuous. Let $y = F_s(x) = |x|^{s-1}x$, then $|y| = |x|^s$ which leads to the inverse map $F_s^{-1} : \mathbb{B}^n \to \mathbb{B}^n,\ x \mapsto |x|^{(1-s)/s}x$. The inverse map is also continuous. Thus, $F_s$ is a homeomorphism. Furthermore, only when $s = 1$, $F_s, F_s^{-1} = \mathrm{id}_{\mathbb{B}^n}$ are smooth and then diffeomorphism, otherwise the negative power will apear up to some order of derivative which breaks the smoothness.

Suppose $\mathcal{A} := \{U_\alpha, \phi_\alpha\}$ is a smooth structure on $M$. Let's consider the composite coordinate chart

$\mathcal{A}_s := \{U_\alpha, \phi_\alpha^{(s)}\}$ where $\phi_\alpha^{(s)} : U_\alpha \to \mathbb{B}^n,\ p \mapsto F_s \circ \phi_\alpha(p)$. Because $F_s$ is a homeomorphism, the composite map $\phi_\alpha^{(s)}$ is again a homeomorphism. It is evident that $\mathcal{A} = \mathcal{A}_1$ because $F_1 = \text{id}$. Then, it needs to show that $\mathcal{A}$ and $\mathcal{A}_s$ are not equivalent when $s \neq 1$. Consider the composite map $\phi_\alpha^{(s)} \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha^{(s)}(U_\alpha \cap U_\beta),\ x \mapsto F_s(\phi_\alpha(\phi_\beta^{-1}(x)))$. Because $F_s$ is not smooth for $s \neq 1$, it means $\phi_\alpha^{(s)}$ and $\phi_\beta$ is not compatible. Then, the union of $\mathcal{A}, \mathcal{A}_s$ is not again an atlas. Thus, $\mathcal{A}$ and $\mathcal{A}_s$ are not equivalent. Since the positive real number is uncountable, we are able to construct uncountably many distinct smooth structures from the given one.

# 3 Problem 2-1

Consider the composite map $F := \psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)$. Because the preimage of $f$ is

$$f^{-1}(V) = \begin{cases} \{x \geqslant 0\} & 1 \in V, 0 \notin V \\ \{x < 0\} & 0 \in V, 1 \notin V \\ \mathbb{R} & 0, 1 \in V \\ \varnothing & 0, 1 \notin V \end{cases} \Rightarrow U \cap f^{-1}(V) = \begin{cases} U \cap \{x \geqslant 0\} & 1 \in V, 0 \notin V \\ U \cap \{x < 0\} & 0 \in V, 1 \notin V \\ U & 0, 1 \in V \\ \varnothing & 0, 1 \notin V \end{cases} \tag{1}$$

If $0, 1$ are both in or not in $V$, the composite maps in both situations are obviously smooth. In other two situations, the composite map is a constant map sending the point on the domain to either $\psi(0)$ or $\psi(1)$. Then, it can be concluded that the composite map is smooth.

However, because $f : \mathbb{R} \to \mathbb{R}$ itself is not continuous at $x = 0$, the composite map $\psi \circ f \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is not smooth. Thus, $f$ is not smooth in the sense of smooth functions on the manifold.

# 4 Problem 2-9

The commutative diagram gives $\widetilde{p} \circ G(z) = \widetilde{p}([z, 1]) = G \circ p(z) = [p(z), 1]$. Because of the equivalent relation $[cz, c] = [z, 1], \forall c \in \mathbb{C}\backslash\{0\}$, it means $\widetilde{p}([z_1, z_2]) = [p(\frac{z_1}{z_2}), 1]$ when $z_2 \neq 0$. Thus, define the map as

$$\widetilde{p}([z_1, z_2]) = \begin{cases} \left[p\left(\frac{z_1}{z_2}\right), 1\right] & , z_2 \neq 0, \\ [1, 0] & , \text{otherwise.} \end{cases} \tag{2}$$

Then, we need to verify that $\widetilde{p} : \mathbb{CP} \to \mathbb{CP}$ defined above is a smooth map. Consider two charts $U_1 = \{[z_1, z_2] \in \mathbb{CP} : z_2 \neq 0\}, \phi_1([z_1, z_2]) = \frac{z_1}{z_2}$ and $U_2 = \{[z_1, z_2] \in \mathbb{CP} : z_1 \neq 0\}, \phi_2([z_1, z_2]) = \frac{z_2}{z_1}$. The composition $\phi_2 \circ \phi_1^{-1} : \mathbb{C} \to \mathbb{C},\ z \mapsto \frac{1}{z}$ is smooth. Thus, those two charts form the smooth structure on $\mathbb{CP}$.

The compositee map $\phi_1 \circ \widetilde{p} \circ \phi_1^{-1} : \mathbb{C} \to \mathbb{C},\ z \mapsto p(z)$ is smooth because $p$ is a polynomial. The composite map $\phi_1 \circ \widetilde{p} \circ \phi_2^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\},\ z \mapsto p(\frac{1}{z})$ is smooth. The composite map $\phi_2 \circ \widetilde{p} \circ \phi_1^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\},\ z \mapsto 1/p(z)$ is smooth. The composite map

$$\phi_2 \circ \widetilde{p} \circ \phi_2^{-1} : \mathbb{C} \to \mathbb{C},\ z \mapsto \begin{cases} 1/p(\frac{1}{z}) & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

is smooth because $1/p(\frac{1}{z})$ can be expressed as a ratio of two polynomials and it is 0 at the origin.

Thus, $\widetilde{p} : \mathbb{C} \to \mathbb{C}$ is the unique smooth continuation of $p : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{p} & \mathbb{C} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbb{CP} & \xrightarrow{\widetilde{p}} & \mathbb{CP} \end{array}$$

# 5 Problem 2-14

According to Theorem 2.29, for closed subsets on the manifold $A, B$, we can find smooth nonnegative functions $f_A, f_B$ such that $f_A^{-1}(0) = A, f_B^{-1}(0) = B$ respectively. Let $f : M \to \mathbb{R},\ p \mapsto f(p) = \frac{f_A(p)}{f_A(p)+f_B(p)}$. Because $A, B$ are disjoint, $f$ is well defined with non-singular denominator. Since both $f_A, f_B$ are nonnegative, $0 \leq f(x) \leq 1$. $f^{-1}(0) = \{p \in M : f_A(p) = 0\} = f_A^{-1}(0) = A$, $f^{-1}(1) = \{p \in M : f_B(p) = 0\} = f_B^{-1}(0) = B$. The smoothness of $f_A, f_B$ implies that of $f$.