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Math 214, Homework 10.

13-16. Suppose $f(t)$ is a smooth positive function on $\mathbb{R}$ such that the improper integral $\int_0^\infty \sqrt{f(t)}dt$ converges. We claim that the sequence $\{n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ equipped with the Riemannian metric $f(t)dt^2$.

Indeed, for $n < m \in \mathbb{R}$, the distance from n to m is bounded by the length of the curve $\gamma : [n, m] \to \mathbb{R} : \gamma(t) = t$ which is

$$\int_n^m |\gamma'(t)|_{f(t)dt^2} dt = \int_n^m \sqrt{f(t)}|\gamma'(t)|dt = \int_n^m \sqrt{f(t)}dt \leq \int_n^\infty \sqrt{f(t)}dt \xrightarrow{n\to\infty} 0$$

So as $n, m \to \infty$, we find that $d_{f(t)dt^2}(n, m) \to 0$, and the sequence is Cauchy.

However, it does not converge. This is because, for any $t \in \mathbb{R}$, once $n > N > t$ we have $d(t, n) \geq d(t, N) > 0$. So the sequence cannot cluster at any point in $\mathbb{R}$.

This shows that $(\mathbb{R}, f(t)dt^2)$ is not complete.

Similarly, if $\int_{-\infty}^0 \sqrt{f(t)}dt$ converges, then $\{-n\}_{n\in\mathbb{N}}$ is Cauchy but not convergent.

So if $(\mathbb{R}, dt^2)$ is complete, then both $\int_0^\infty \sqrt{f(t)}dt$ and $\int_{-\infty}^0 \sqrt{f(t)}dt$ diverge.

Conversely, suppose both $\int_0^\infty \sqrt{f(t)}dt$ and $\int_{-\infty}^0 \sqrt{f(t)}dt$ diverge. Then if $\{x_n\}$ is a Cauchy sequence, it is bounded for $d_{f(t)dt^2}$. So it is contained in a set $[-r, r]$ (if it weren't, it would have a subsequence tending to $\pm\infty$, and by divergence of the integrals the subsequence's distance from x_1 would be $\left|\int_{x_1}^{x_{n_k}} \sqrt{f(t)}dt\right|$ which is unbounded).

Such sets are compact for the metric topology, because the metric topology is the same as the original manifold topology on $\mathbb{R}$ (by Theorem 13.29). Every compact metric space is complete, so $\{x_n\}$ converges in $[-r, r]$ for the restricted metric.

Then $(\mathbb{R}, f(t)dt^2)$ is complete.

13-19. Define $F : \mathbb{R} \to \mathbb{R}^2$ by $F(t) = ((e^t + 1)\cos t, (e^t + 1)\sin t)$.

Then F is injective: if $F(s) = F(t)$, we have $|F(s)| = |F(t)|$ so $e^s + 1 = e^t + 1$ and $s = t$.

Also, F has Jacobian $\begin{pmatrix} e^t \cos t - (e^t + 1)\sin t \\ e^t \sin t + (e^t + 1)\cos t \end{pmatrix}$. The vectors $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ and $\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ are always linearly independent, since the determinant of the matrix containing them is 1. Thus F's Jacobian is nonvanishing and F is an immersion.

F has an inverse map from its image, given by $F^{-1}(x, y) = \log(\sqrt{x^2 + y^2} - 1)$.

Denote by $\overline{B(0, R)}$ the closed ball about 0 of radius R in $\mathbb{R}^2$.

Since the image of F is contained in $\mathbb{R}^2 \setminus \overline{B(0, 1)}$, F^{-1} is continuous on its image and F is a homeomorphism onto its image. This shows that F is an embedding.

However, F is not proper because $F^{-1}(\overline{B(0, 2)}) = (-\infty, 0]$ which is noncompact.

Yet we may compute that the induced metric $F^*(dx^2 + dy^2)$ is

$$d((e^t+1)\cos t)^2 + d((e^t+1)\sin t)^2 = \left[(e^t \cos t - (e^t + 1)\sin t)^2 + (e^t \sin t + (e^t + 1)\cos t)^2\right] dt^2 =$$

$$= \left[e^{2t}(\cos^2 t + \sin^2 t) + (e^t + 1)^2(\cos^2 t + \sin^2 t\right] dt^2 = \left[2e^{2t} + 2e^t + 1\right] dt^2$$

Let $f(t) = 2e^{2t} + 2e^t + 1$. Then f is bounded below by 1,

so the improper integrals $\int_{-\infty}^{0} \sqrt{f(t)}dt$ and $\int_0^{\infty} \sqrt{f(t)}dt$ both diverge.

By problem 13-16, this shows that the induced metric on $\mathbb{R}$ is in fact complete.

13-20. $g_1 = F_1^*(dx^2 + dy^2 + dz^2) = du^2 + dv^2 + d(0)^2 = du^2 + dv^2$,
which is the Euclidean metric on $\mathbb{R}^2$. Hence $(\mathbb{R}^2, g_1)$ is not bounded, it is complete, and it is flat, as it is exactly $\mathbb{R}^2$ equipped with the normal Euclidean metric.

$g_2 = F_2^*(dx^2 + dy^2 + dz^2) = du^2 + d(e^v)^2 + d(0)^2 = du^2 + e^{2v}dv^2$.
g_2 is not bounded because the distance from $(0,0)$ to $(0,n)$ satisfies the following bound:
If $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(0) = (0,0)$, $\gamma(1) = (0,n)$, $\gamma = (\gamma_1, \gamma_2)$, then

$$\int_0^1 |\gamma'(t)|_{g_2} dt = \int_0^1 \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2 e^{2\gamma_2(t)}} dt \geq \int_0^1 \gamma_2'(t) e^{\gamma_2(t)} dt = e^n - 1$$

so the distances $d_{g_2}((0,0),(0,n))$ form an unbounded sequence.
Similarly, g_2 is not complete since the sequence $\{(0,-n)\}$ is Cauchy but certainly does not converge in the topology on $\mathbb{R}^2$. To see that it is Cauchy, check:

$$d((0,-n),(0,-m)) \leq \int_{-n}^{-m} e^t dt = e^{-m} - e^{-n} \xrightarrow{n,m\to\infty} 0$$

Finally, g_2 is flat since the function $H : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 : (x,y) \mapsto (x, \log y)$ is locally a diffeomorphism, and $H^* g_2 = dx^2 + e^{2\log y} d(\log y)^2 = dx^2 + y^2 \frac{1}{y^2} dy^2 = dx^2 + dy^2$.

$g_3 = F_3^*(dx^2 + dy^2 + dz^2) = du^2 + dv^2 + d(u^2 + v^2)^2 = (1 + 4u^2)du^2 + (1 + 4v^2)dv^2$
Note that the distance between two points in this metric is always greater than the distance between those points in the Euclidean metric, since $1+4u^2 \geq 1$ and $1+4v^2 \geq 1$:

$$L_{g_3}(\gamma) = \int_0^1 \sqrt{\gamma_1'(t)^2(1 + \gamma_1(t)^2) + \gamma_2'(t)^2)(1 + \gamma_2(t)^2)} dt \geq \int_0^1 \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2} dt = L_{\overline{g}}(\gamma)$$

So $(\mathbb{R}^2, g_3)$ is unbounded. And by the same reasoning, any Cauchy sequence in $(\mathbb{R}^2, g_2)$ is a Cauchy sequence in $(\mathbb{R}^2, \overline{g})$, hence converges in the metric topology for $\overline{g}$, which is the standard topology for $\mathbb{R}^2$, which is the metric topology for g_3.
So any Cauchy sequence converges and $(\mathbb{R}^2, g_3)$ is complete.
However, $(\mathbb{R}^2, g_3)$ is not flat. Indeed, using polar coordinates on R^2 we find that $F_3(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$ is an embedding of a surface of revolution in $\mathbb{R}^3$ generated by the curve $F(r,0) = (r, 0, r^2)$. This curve is not part of a straight line. Then by Prop 13.19, the induced metric is not flat, so neither is g_3 (of course, pulling back, the function F_3 is an isometry between $(\mathbb{R}^2, g_3)$ and $F(R^2)$ with the induced metric).

Note that F_4 is the stereographic embedding of $\mathbb{R}^2$ as $\mathbb{S}^2 \setminus N$ where $N = (0, 0, 1)$.

Then for any two points $p, q \in \mathbb{R}^2$, the distance $d_{g_4}(p, q)$ is bounded by the length of the pullback of the great circle containing $F_4(p)$ and $F_4(q)$, which is of course 2π.
So $(\mathbb{R}^2, g_4)$ is bounded.

It is not complete. The sequence $\{(0, n)\}$ does not converge, yet is Cauchy since $F(0, n) = (0, 2n/(n^2 + 1), (n^2 - 1)/(n^2 + 1))$. Such points have great circle distances $1 - \arccos(2n/(n^2 + 1))$ from N, which converge to 0 as $n \to \infty$. So on $\mathbb{S}^2 \setminus N$ with the induced metric, the sequence is Cauchy. Hence $(\mathbb{R}^2, g_4)$ is not complete.

Finally, as before it is not flat because $F_4(\mathbb{R}^2)$ is a surface of revolution and by using polar coordinates on $\mathbb{R}^2$ we find that F_4 is an isometry between $(\mathbb{R}^2, g_4)$ and the surface of revolution of the curve $\{(2r/(r^2+1), 0, (r^2-1)/(r^2+1)) : r \geq 0\}$ which is a semicircle with the north endpoint missing. By Prop 13.19, this curve is not part of a straight line and hence g_4 is not flat.

4. If $M = \mathbb{R}^2$, L is the trivial line bundle on M, and $\nabla = d + A$ where d is the trivial connection on L and $A = xdy - ydx$. Let $a = (1, 0)$, $b = (-1, 0)$, and $\gamma_\pm : [0, \pi] \to \mathbb{R}^2$ be paths from a to b with $\gamma_\pm(t) = (\cos t, \pm \sin t)$.

Since the bundle is trivial, we may compute the parallel transport from a to b using the parallel transport ODE. Firstly, if $e_1 \in \Omega^0(M, L)$ is the constant section $e_1(x, y) = 1$, then $\nabla(e_1) = d(e_1) + Ae_1 = -ydx + xdy$, so $\Gamma^1_{11}(x, y) = -y$ and $\Gamma^1_{21}(x, y) = x$.
If $u_0 \in L_a$ and we define $u(t) \in E_{\gamma+(t)}$ by the following ODE:

$$\begin{cases} \dfrac{du(t)}{dt} + \left[\Gamma^1_{11}(\gamma_+(t))\dot{\gamma}^1_+(t) + \Gamma^1_{21}(\gamma_+(t))\dot{\gamma}^2_+(t)\right] u(t) = 0 \\ u(0) = u_0 \end{cases}$$

Substituting in, we find: $\dfrac{du(t)}{dt} + (\sin^2 t + \cos^2 t)u(t) = \dfrac{du(t)}{dt} + u(t) = 0$ so $u(t) = u_0 e^{-t}$.
Then $u(\pi) = u_0 e^{-\pi}$, and we find that P_{γ_+} is the map $v \mapsto e^{-\pi} v$.

Similarly, if $u_0 \in L_a$ and we define $u(t) \in E_{\gamma-(t)}$ by the following ODE:

$$\begin{cases} \dfrac{du(t)}{dt} + \left[\Gamma^1_{11}(\gamma_-(t))\dot{\gamma}^1_-(t) + \Gamma^1_{21}(\gamma_-(t))\dot{\gamma}^2_-(t)\right] u(t) = 0 \\ u(0) = u_0 \end{cases}$$

Substituting in, we find: $\dfrac{du(t)}{dt} + (-\sin^2 t - \cos^2 t)u(t) = \dfrac{du(t)}{dt} - u(t) = 0$ so $u(t) = u_0 e^{t}$.
Then $u(\pi) = u_0 e^{\pi}$, and we find that P_{γ_-} is the map $v \mapsto e^{\pi} v$.

5. Let $M = \mathbb{R}^2$, E the trivial rank-2 vector bundle on M, and $\nabla = d + A$ where d is the trivial connection and $A \in \Omega^1(M, \text{End}(E))$ is the connection 1-form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} dy$$

Then the symbols Γ_1 and Γ_2 are $\Gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\Gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and by our local coordinate formulas for the curvature 2-form $F = dx^i \wedge dx^j \otimes F_{ij}$ where $F_{ij} = \partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = [\Gamma_i, \Gamma_j]$ for our constant matrices.

Then $F_{11} = F_{22} = 0$, $F_{12} = -F_{21} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Thus $F = dx \wedge dy \otimes \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

To see this another way, recall that $F = dA + A \wedge A$.

$dA = d(dx \otimes \Gamma_1) + d(dy \otimes \Gamma_2) = d(dx) \otimes \Gamma_1 + d(dy) \otimes \Gamma_2 = 0$ since Γ_1, Γ_2 are constant.

So $F = A \wedge A = dx \wedge dy \otimes \Gamma_1 \Gamma_2 + dy \wedge dx \otimes \Gamma_2 \Gamma_1 = dx \wedge dy \otimes [\Gamma_1, \Gamma_2] = dx \wedge dy \otimes \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.