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Math 214, Homework 11.

1. Let $M = S^2$ embedded in $\mathbb{R}^3$ with spherical coordinates $(\theta, \varphi) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ so that the induced metric is given by:

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= d(\sin\theta\cos\varphi)^2 + d(\sin\theta\sin\varphi)^2 + d(\cos\theta)^2 \\ &= (\cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi)^2 + (\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi)^2 + \sin^2\theta d\theta^2 \\ &= d\theta^2 + \sin^2\theta\, d\varphi^2 \end{aligned}$$

Then we need to compute the Levi-Cevita connection for this metric. The symbols are

$$\begin{aligned} \Gamma^1_{11} &= g^{11}(-\partial_\theta g_{11} + \partial_\theta g_{11} + \partial_\theta g_{11})/2 = 0 \\ \Gamma^2_{11} &= g^{22}(-\partial_\varphi g_{11} + \partial_\theta g_{12} + \partial_\theta g_{12})/2 = 0 \\ \Gamma^1_{12} = \Gamma^1_{21} &= g^{11}(-\partial_\theta g_{12} + \partial_\theta g_{21} + \partial_\varphi g_{11})/2 = 0 \\ \Gamma^2_{12} = \Gamma^2_{21} &= g^{22}(-\partial_\varphi g_{12} + \partial_\theta g_{22} + \partial_\varphi g_{12})/2 = \cot\theta \\ \Gamma^1_{22} &= g^{11}(-\partial_\theta g_{22} + \partial_\varphi g_{12} + \partial_\varphi g_{21})/2 = -\sin\theta\cos\theta \\ \Gamma^2_{22} &= g^{22}(-\partial_\varphi g_{22} + \partial_\varphi g_{22} + \partial_\varphi g_{22})/2 = 0 \end{aligned}$$

Now, let $\theta_0 \in (0, \pi)$, $\gamma : [0, 1] \to S^2$ be the curve $\gamma(t) = (\theta_0, 2\pi t)$, and $u_0 = \partial_\varphi \in T_{\gamma(0)}S^2$. The parallel transport of u_0 by γ is given by the ODEs

$$\begin{cases} \frac{du^1}{dt} + \Gamma^1_{jk}(\gamma(t))\dot\gamma^j(t)u^k(t) = 0 \\ \frac{du^2}{dt} + \Gamma^2_{jk}(\gamma(t))\dot\gamma^j(t)u^k(t) = 0 \\ u^1(0) = 0, \quad u^2(0) = 1 \end{cases}$$

Since $\dot\gamma^1(t) = 0$ and $\dot\gamma^2(t) = 2\pi$ for all t, this becomes the system

$$\begin{cases} \frac{du^1}{dt} - (2\pi\sin\theta_0\cos\theta_0)u^2(t) = 0 \\ \frac{du^2}{dt} + (2\pi\cot\theta_0)u^1(t) = 0 \\ u^1(0) = 0, \quad u^2(0) = 1 \end{cases}$$

Then $\frac{d^2u^1}{dt^2} + (4\pi^2\cos^2\theta_0)u^1(t) = 0$ and $u^1(t) = C\sin((2\pi\cos\theta_0)t)$.
Then $u^2(t) = C(\csc\theta_0)\cos((2\pi\cos\theta_0)t)$, $C = \sin\theta_0$, and we find:

$$u(t) = ((\sin\theta_0)\sin((2\pi\cos\theta_0)t), \cos(2\pi\cos\theta_0)t))$$

We find that the desired parallel transport of u_0 is

$$u(1) = \sin(\theta_0)\sin(2\pi\cos\theta_0)\partial_\theta + \cos(2\pi\cos\theta_0)\partial_\varphi \in T_{\gamma(1)}S^2 = T_{\gamma(0)}S^2$$

2. For $\delta > 0$, let M be the submanifold given by the embedding $(0, 2\pi] \times (-\delta, \delta) \to \mathbb{R}^3$, mapping $(\varphi, z) \mapsto ((1 + z\sin(\varphi/2))\cos\varphi, ((1 + z\sin(\varphi/2))\sin\varphi, z\cos(\varphi/2))$.

   Then the induced metric from $\mathbb{R}^3$ is given by:

$$\begin{aligned}
\sum_i (dx^i)^2 &= d((1 + z\sin(\varphi/2))\cos\varphi)^2 + d(((1 + z\sin(\varphi/2))\sin\varphi)^2 + d(z\cos(\varphi/2))^2 \\
&= (\sin(\varphi/2)\cos\varphi\, dz + (z/2\cos(\varphi/2)\cos\varphi - (1 + z\sin(\varphi/2))\sin\varphi)\, d\varphi)^2 \\
&\quad + (\sin(\varphi/2)\sin\varphi\, dz + (z/2\cos(\varphi/2)\sin\varphi + (1 + z\sin(\varphi/2))\cos\varphi)\, d\varphi)^2 \\
&\quad + (\cos(\varphi/2)\, dz - z/2\sin(\varphi/2)\, d\varphi)^2 \\
&= dz^2 + (z^2/4 + (1 + z\sin(\varphi/2))^2)d\varphi^2
\end{aligned}$$

   Note that since $g_{zz} = 1$, $g_{z\varphi} = 0$, we immediately have $\Gamma^z_{zz} = \Gamma^z_{z\varphi} = \Gamma^z_{\varphi z} = \Gamma^\varphi_{zz} = 0$.

   Then we can compute the Riemann curvature tensor component $R^z_{\varphi\varphi z}$:

   $R^z_{\varphi\varphi z} = \partial_\varphi \Gamma^z_{z\varphi} - \partial_z \Gamma^z_{\varphi\varphi} + \Gamma^z_{z\varphi}\Gamma^z_{z\varphi} + \Gamma^z_{\varphi\varphi}\Gamma^\varphi_{z\varphi} - \Gamma^z_{zz}\Gamma^z_{\varphi\varphi} - \Gamma^z_{z\varphi}\Gamma^\varphi_{\varphi\varphi} = -\partial_z \Gamma^z_{\varphi\varphi} + \Gamma^z_{\varphi\varphi}\Gamma^\varphi_{z\varphi}$

   We compute the Christoffel symbols:

   $\Gamma^z_{\varphi\varphi} = \frac{1}{2}g^{zz}(-\partial_z g_{\varphi\varphi}) = -\frac{1}{2}\partial_z(z^2/4 + (1+z\sin(\varphi/2))^2) = -z/4 - (1+z\sin(\varphi/2))\sin(\varphi/2)$.

   $\Gamma^\varphi_{z\varphi} = \frac{1}{2}g^{\varphi\varphi}(\partial_z g_{\varphi\varphi}) = \dfrac{z/4 + (1 + z\sin(\varphi/2))\sin(\varphi/2)}{z^2/4 + (1 + z\sin(\varphi/2))^2}$

   Then we find that

   $R^z_{\varphi\varphi z} = 1/4 + \sin^2(\varphi/2) - \dfrac{(z/4 + (1 + z\sin(\varphi/2))\sin(\varphi/2))^2}{z^2/4 + (1 + z\sin(\varphi/2))^2}$

   which, when $z = \varphi = 0$, is $1/4$.

   Hence the curvature does not vanish everywhere and the metric is not flat.

3. We proceed similarly as in the last problem.

   For $\delta > 0$, let M be the submanifold given by the embedding $\mathbb{R} \times (-\delta, \delta) \to \mathbb{R}^3$, mapping $(t, z) \mapsto (z\cos(\sin t), z\sin(\sin t), t)$.

   Then the induced metric from $\mathbb{R}^3$ is given by:

$$\begin{aligned}
\sum_i (dx^i)^2 &= d(z\cos(\sin t))^2 + d(z\sin(\sin t))^2 + dt^2 \\
&= (\cos(\sin t)\,dz - z\sin(\sin t)\cos t\,dt)^2 + (\sin(\sin t)\,dz + z\cos(\sin t)\cos t\,dt)^2 + dt^2 \\
&= dz^2 + (1 + z^2\cos^2 t)\,dt^2
\end{aligned}$$

   Note that since $g_{zz} = 1$, $g_{zt} = 0$, we immediately have $\Gamma^z_{zz} = \Gamma^z_{zt} = \Gamma^z_{tz} = \Gamma^t_{zz} = 0$.

   Then we can compute the Riemann curvature tensor component R^z_{ttz}:

   $R^z_{ttz} = \partial_t \Gamma^z_{zt} - \partial_z \Gamma^z_{tt} + \Gamma^z_{zt}\Gamma^z_{zt} + \Gamma^z_{tt}\Gamma^t_{zt} - \Gamma^z_{zz}\Gamma^z_{tt} - \Gamma^z_{zt}\Gamma^t_{tt} = -\partial_z \Gamma^z_{tt} + \Gamma^z_{tt}\Gamma^t_{zt}$

   We compute the Christoffel symbols:

   $\Gamma^z_{tt} = \frac{1}{2} g^{zz}(-\partial_z g_{tt}) = -z\cos^2 t$

   $\Gamma^t_{zt} = \frac{1}{2} g^{tt}(\partial_z g_{tt}) = \dfrac{z\cos^2 t}{1 + z^2\cos^2 t}$

   Then we find that

   $R^z_{ttz} = \cos^2 t - \dfrac{z^2\cos^4 t}{1 + z^2\cos^2 t} = \dfrac{\cos^2 t}{1 + z^2\cos^2 t}$

   which, when $z = t = 0$, is 1.

   Hence the curvature does not vanish everywhere, and the metric is not flat.

4. Let G be a Lie group, $X, Y \in T_eG$. Consider the curve $\gamma(t) = \exp(tY)$, which is both the geodesic through e in direction Y and the integral curve through e of the left-invariant vector field generated by Y (Nic 4.1.15).

Recall the parallel transport equation, $\nabla_{\dot\gamma(t)} u(t) = 0$.

Let $u(t) := (L_{\gamma(t/2)})_* (R_{\gamma(t/2)})_* X$.

We have that $\dot\gamma(t)$ is the restriction of the left-invariant v.f. Z generated by Y to γ.

Let X_i be a global left-invariant frame for TG, and write $u(t) = U^i(t) X_i|_{\gamma(t)}$.

Then $\nabla_{\dot\gamma(t)}(u(t)) = \sum_i \frac{d}{dt} U^i(t) X_i|_{\gamma(t)} + \sum_i U^i(t)(\nabla_Z X_i)|_{\gamma(t)}$.

We have $\nabla_Z X_i = \frac{1}{2}[Z, X_i]$, by our formula for the Levi-Cevita connection on G.

We also have the following infinitessimal change of u in terms of the Ad representation:

$$u(t+\varepsilon) = (L_{\gamma((t+\varepsilon)/2)})_* (R_{\gamma((t+\varepsilon)/2)})_* X = (R_{\gamma(\varepsilon/2)})_* (L_{\gamma(\varepsilon/2)})_* u(t) =$$

$$= (R_{\gamma(\varepsilon/2)})_* (L_{\gamma(-\varepsilon/2)})_* (L_{\gamma(\varepsilon)})_* u(t) = \mathrm{Ad}_{\gamma(-\varepsilon/2)}((L_{\gamma(\varepsilon)})_* u(t))$$

(where we have exploited the fact that L_g and R_g commute).

So, noting that the X_i are invariant with change of time, we can write

$$\sum_i \frac{d}{dt} U^i(t) X_i|_{\gamma(t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(u(t+\varepsilon) - (L_{\gamma(\varepsilon)})_* u(t)) =$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon}(\mathrm{Ad}_{\gamma(-\varepsilon/2)} - I) \right] (L_{\gamma(\varepsilon)})_* u(t) = \mathrm{ad}_{-\frac{1}{2}Y} u(t) = \left[-\frac{1}{2} Z, u(t) \right] = -\frac{1}{2} U^i(t)[Z, X_i]$$

where we've taken that the differential of Ad_e is ad (which is acting on $u(t)$, t constant, as though it's part of a left-invariant vector field: it's acting on $U^i(t)X_i$ with t fixed).

(If it's not kosher to assume this, here's a short proof: Since R_g and L_h commute, the flow of a left invariant vector field commutes with R_g, i.e. $\varphi_t(g) = R_{\varphi_t(e)} g$, and we can go by the Lie derivative:

$$[-1/2Z, u(t)] = \nabla_{-1/2Z} u(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\exp(-\varepsilon/2Z)_* u(t) - u(t)) =$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}((R_{\exp(-\varepsilon/2Z)})_* u(t) - u(t)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}((R_{\exp(-\varepsilon/2Z)})_* (L_{\exp(-\varepsilon/2Z)})_* u(t) - u(t))$$

which is where we started... Oh dear, this is a mess).

So we find that the parallel transport equation is satisfied:

$$\nabla_{\dot\gamma(t)}(u(t)) = \sum_i \left(U^i(t)(\nabla_Z X_i)|_{\gamma(t)} + \frac{d}{dt} U^i(t) X_i|_{\gamma(t)} \right) = \frac{1}{2} U^i(t)[Z, X_i] - \frac{1}{2} U^i(t)\,[Z, X_i] = 0$$

Since solutions to ODE are unique, $u(t)$ is indeed the parallel transport of X along $\gamma(t)$.

5. First of all, $SU(2) = \{A \in M(2,\mathbb{C}) : A^*A = I, \det A = 1\}$,

so $su(2) = T_I(SU(2)) = \ker dF_I \cap \ker dG_I$ where $F : GL(2,\mathbb{C}) \to GL(2,\mathbb{C})$, $F(A) = A^*A$, and $G : GL(2,\mathbb{C}) \to \mathbb{R}$, $G(A) = \det A$.

We showed in the past that $dF_I A = A + A^*$ and $dG_I A = \mathrm{tr}(A)$, so $su(2) = \{A \in M(2,\mathbb{C}) : A = -A^*, \mathrm{tr}(A) = 0\}$.

Then $su(2)$ as a real vector space has as a basis the matrices $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\}$,

which we denote E_1, E_2, E_3. Then for $a, b \in \mathbb{R}$, $z = z_1 + iz_2, w = w_1 + iw_2 \in \mathbb{C}$

we let $X = \begin{pmatrix} ai & z \\ -\overline{z} & -ai \end{pmatrix}, Y = \begin{pmatrix} bi & w \\ -\overline{w} & -bi \end{pmatrix} \in su(2)$ and compute $\mathrm{ad}(X)\mathrm{ad}(Y)$ on E_i:

$$\mathrm{ad}(X)\mathrm{ad}(Y)E_1 = \left[X, \left[Y, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\right] = \left[X, \begin{pmatrix} -2w_2 i & 2bi \\ 2bi & 2w_2 i \end{pmatrix}\right] = \begin{pmatrix} 4bz_1 i & -4ab + 4w_2 zi \\ 4ab + 4w_2\overline{z}i & -4bz_1 i \end{pmatrix}$$

so that the E_1 coordinate is $\mathrm{Re}(-4ab + 4w_2 zi) = -4(ab + z_2 w_2)$,

$$\mathrm{ad}(X)\mathrm{ad}(Y)E_2 = \left[X, \left[Y, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right]\right] = \left[X, \begin{pmatrix} 2w_1 i & -2b \\ 2b & -2w_1 i \end{pmatrix}\right] = \begin{pmatrix} * & -4abi - 4w_1 zi \\ -4abi - 4w_1\overline{z}i & * \end{pmatrix}$$

so that the E_2 coordinate is $\mathrm{Im}(-4abi - 4w_1 zi) = -4(ab + z_1 w_1)$,

$$\mathrm{ad}(X)\mathrm{ad}(Y)E_3 = \left[X, \left[Y, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right]\right] = \left[X, \begin{pmatrix} 0 & -2iw \\ -2i\overline{w} & 0 \end{pmatrix}\right] = \begin{pmatrix} -2(\overline{z}w + z\overline{w})i & * \\ * & 2(\overline{z}w + z\overline{w})i \end{pmatrix}$$

so that the E_3 coordinate is $-2(\overline{z}w + z\overline{w}) = -4\mathrm{Re}(\overline{z}w) = -4(z_1 w_1 + z_2 w_2)$.

Hence the Killing form is given by $\kappa(X, Y) = -\mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) =$
$= -(-4(ab + z_2 w_2) + -4(ab + z_1 w_1) + -4(z_1 w_1 + z_2 w_2)) = 8(ab + z_1 w_1 + z_2 w_2)$.

One could write this as a matrix, $\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

Computation for $sl(2,\mathbb{R})$ on next page.

First of all, $SL(2,\mathbb{R}) = \{A \in M(2,\mathbb{R}) : \det A = 1\}$.
Then $sl(2,\mathbb{R}) = \ker d(\det)_I = \{A \in M(2,\mathbb{R}) : \mathrm{tr}(A) = 0\}$.
We have a basis of matrices $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ which we call F_1, F_2, F_3.

Then for $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, letting $X = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}, Y = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}$,

we compute $\mathrm{ad}(X)\mathrm{ad}(Y)$ on F_i:

$$\mathrm{ad}(X)\mathrm{ad}(Y)F_1 = \left[X, \left[Y, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\right] = \left[X, \begin{pmatrix} 0 & -2b_2 \\ 2b_3 & 0 \end{pmatrix}\right] = \begin{pmatrix} 2a_2b_3 + 2a_3b_2 & * \\ * & -2a_2b_3 - 2a_3b_2 \end{pmatrix}$$

so that the F_1 coordinate is $2a_2b_3 + 2a_3b_2$,

$$\mathrm{ad}(X)\mathrm{ad}(Y)F_2 = \left[X, \left[Y, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]\right] = \left[X, \begin{pmatrix} -b_3 & 2b_1 \\ 0 & b_3 \end{pmatrix}\right] = \begin{pmatrix} * & 4a_1b_1 + 2a_2b_3 \\ * & * \end{pmatrix}$$

so that the F_2 coordinate is $4a_1b_1 + 2a_2b_3$,

$$\mathrm{ad}(X)\mathrm{ad}(Y)F_3 = \left[X, \left[Y, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]\right] = \left[X, \begin{pmatrix} b_2 & 0 \\ -2b_1 & -b_2 \end{pmatrix}\right] = \begin{pmatrix} * & * \\ 4a_1b_1 + 2a_3b_2 & * \end{pmatrix}$$

so that the F_3 coordinate is $4a_1b_1 + 2a_2b_3$.

Then the Killing form is given by $\kappa(X,Y) = -\mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) =$
$= -(2a_2b_3 + 2a_3b_2 + 4a_1b_1 + 2a_2b_3 + 4a_1b_1 + 2a_2b_3) = -8a_1b_1 - 4a_2b_3 - 4a_3b_2$.

One could write this as a matrix, $\begin{pmatrix} -8 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{pmatrix}$.

(Note that $SL(2,\mathbb{R})$ is not compact, and that this form is not positive semidefinite).