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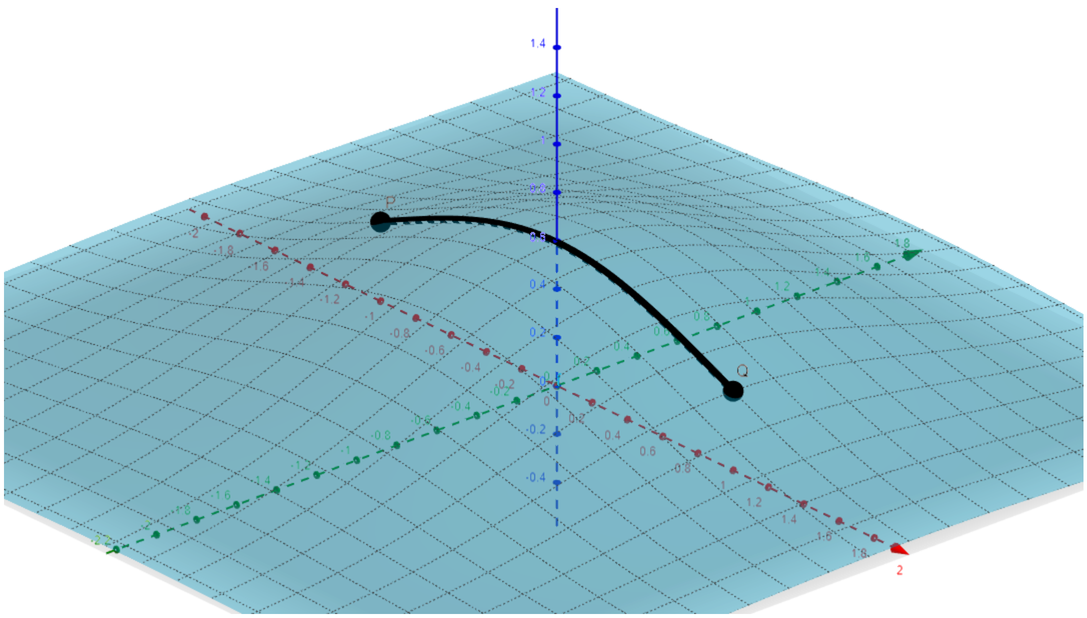
Math 214, Homework 12.

1. (i) As $r \to \infty$, the functions h_r and their differentials uniformly approach 0 so that the induced metric is a small perturbation of $dx^2 + dy^2$.

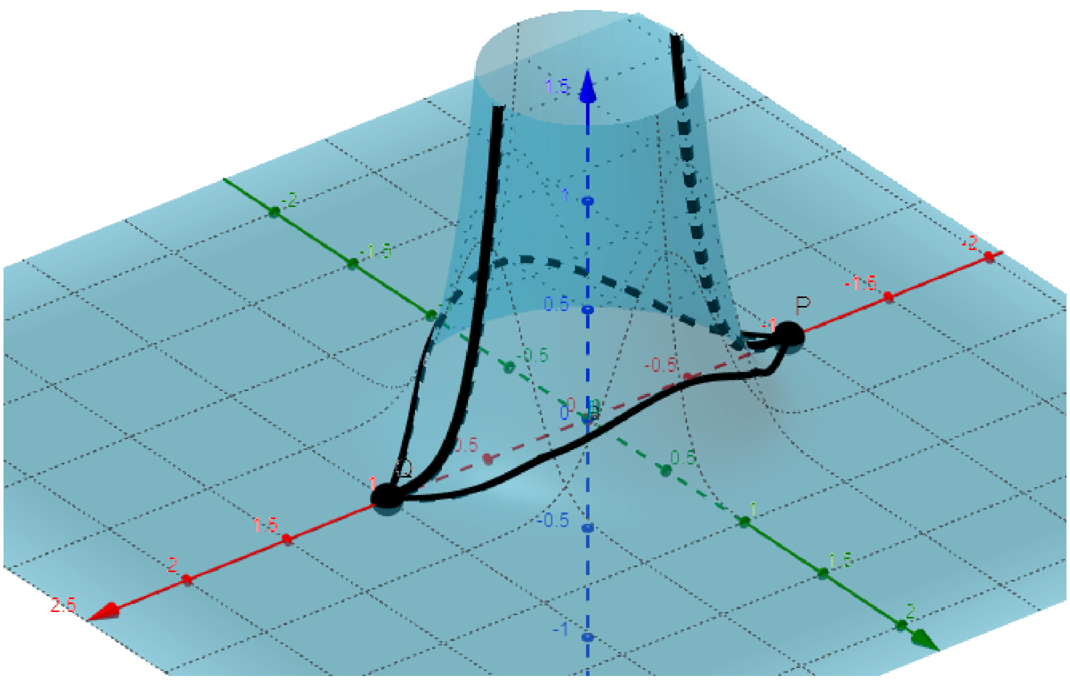
   Then for large enough r, even $r > 5$, there is only one geodesic connecting p to q, the horizontal straight line path from p to q.



(ii) There is an r at which the single geodesic between p and q bifurcates into 3 geodesics, the same straight line, as well as a geodesic moving left of the peak and a geodesic moving right of the peak.

From there, as $r \to 0$, the number of geodesics connecting p and q diverges to infinity. These geodesics are curves which originate from p, wind up the peak before hitting the y-axis at a perpendicular angle, and then wind back down the peak to hit q. For any given r, there are finitely many such geodesics because the geodesic ODE is nice enough to exhibit continuous dependence on initial conditions, which prevents occurrences of "infinite wrapping" leading to infinitely fast rotation of the exit trajectory of the geodesic off the hill. Note that we have not shown any geodesics which wrap around the hill multiple times in the picture.

But if we attempt to take a "limiting metric" and get a manifold with connected component $\mathbb{R}^2 \setminus \{(0,0)\}$, we find that there are no geodesics connecting p and q.



(iii) For most finite r, there are only finitely many geodesics between p and q. Consider the case of $r = 10$. Then any path besides the straight line path connecting p and q is not length extremizing locally, because one can look at a point where the path has deviated from being a straight line and straighten it out to decrease the distance (dh_{10} is very small so has little effect on the length) or further perturb it outward to increase the distance. See the image for (i) where $r = 2$.
Hence there is only one geodesic connecting p and q when $r = 10$.

2. If $f(x,y) = x^2 - y^2$ and we consider the embedded submanifold $S \subset \mathbb{R}^3$ given by the graph of f, we can compute the second fundamental form at $(0,0)$.

First of all, S is the image of the embedding $F(x,y) = (x, y, f(x,y))$,
so $dF_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ which has image $\mathbb{R}^2 \times \{0\}$.
So the normal space to S at $(0,0)$ is spanned by the vector $n = \partial_z$.

Then we have the trivial connection d on $\mathbb{R}^3$. The two coordinate vector fields on S are given by $\partial_1 = \partial_x + 2x\partial_z$ and $\partial_2 = \partial_y - 2y\partial_z$ written in $\mathbb{R}^3$, and we compute $d_{\partial_i}\partial_j$:

$d_{\partial_1}\partial_1 = d_{\partial_x}\partial_x + 2xd_{\partial_z}\partial_x + d_{\partial_x}(2x\partial_z) + 2xd_{\partial_z}(2x\partial_z) = 2\partial_z$

$d_{\partial_1}\partial_2 = d_{\partial_x}\partial_y + 2xd_{\partial_z}\partial_y + d_{\partial_x}(-2y\partial_z) + 2xd_{\partial_z}(-2y\partial_z) = 0$

$d_{\partial_2}\partial_1 = d_{\partial_y}\partial_x - 2yd_{\partial_z}\partial_x + d_{\partial_y}(2x\partial_z) - 2yd_{\partial_z}(2x\partial_z) = 0$

$d_{\partial_2}\partial_2 = d_{\partial_y}\partial_y - 2yd_{\partial_z}\partial_y + d_{\partial_y}(-2y\partial_z) - 2yd_{\partial_z}(-2y\partial_z) = -2\partial_z$

Then at $(0,0)$, $\mathcal{N}(\partial_i, \partial_j) = \text{Proj}_{N_{(0,0)}S}(d_{\partial_i}\partial_j) = \langle d_{\partial_i}\partial_j, n\rangle\, n$ is given by:

$\mathcal{N}(\partial_x, \partial_x) = \langle 2\partial_z, \partial_z\rangle\, \partial_z = 2\partial_z,$

$\mathcal{N}(\partial_x, \partial_y) = \mathcal{N}(\partial_y, \partial_x) = 0,$

$\mathcal{N}(\partial_y, \partial_y) = \langle -2\partial_z, \partial_z\rangle\, \partial_z = -2\partial_z,$

or by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \partial_z$ in the $\{\partial_x, \partial_y\}$ basis for $T_{(0,0)}S$.

3. Let G be a compact Lie group with bi-invariant metric $\langle \cdot, \cdot \rangle$. Then for X, Y, Z left-invariant vector fields, recall that $\nabla_X Y = \frac{1}{2}[X, Y]$ for the Levi-Cevita connection.

   Recall that $[X, Y]$ is also left-invariant. Then we compute the Riemannian curvature:

$$
\begin{aligned}
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= \nabla_X \left(\frac{1}{2}[Y,Z] \right) - \nabla_Y \left(\frac{1}{2}[X,Z] \right) - \frac{1}{2}[[X,Y],Z] \\
&= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\
&= -\frac{1}{4}[[Y,Z],X] - \frac{1}{4}[[Z,X],Y] - \frac{1}{4}[[X,Y],Z] - \frac{1}{4}[[X,Y],Z] \\
&= -\frac{1}{4}[[X,Y],Z]
\end{aligned}
$$

4. When X, Y, Z are left-invariant vector fields on compact Lie group G, $\langle \cdot, \cdot \rangle$ bi-invariant, the exterior derivative of the form $B(X, Y, Z) = \langle [X, Y], Z \rangle$ can be computed by Prop. 14.32 in Lee. First, note that $[X, Y]$ is also left-invariant, so $\langle [X, Y], Z \rangle$ is a constant. Let W be another left-invariant vector field. Then we compute:

$$\begin{aligned}(dB)(X,Y,Z,W) &= X(B(Y,Z,W)) - Y(B(X,Z,W)) + Z(B(X,Y,W)) - W(B(X,Y,Z))\\ &\quad - B([X,Y],Z,W) + B([X,Z],Y,W) - B([X,W],Y,Z)\\ &\quad - B([Y,Z],X,W) + B([Y,W],X,Z) - B([Z,W],X,Y)\end{aligned}$$

$$\begin{aligned}&= 0 - 0 + 0 - 0 - \langle [[X,Y],Z],W\rangle + \langle [[X,Z],Y],W\rangle - \langle [[X,W],Y],Z\rangle\\ &\qquad - \langle [[Y,Z],X],W\rangle + \langle [[Y,W],X],Z\rangle - \langle [[Z,W],X],Y\rangle\\ &= -\langle [[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X], W\rangle\\ &\quad + \langle [[W,X],Y] + [[Y,W],X], Z\rangle - \langle [[Z,W],X],Y\rangle\\ &= 0 - \langle [[X,Y],W],Z\rangle + \langle [[W,Z],X],Y\rangle\\ &= \langle [X,Y],[W,Z]\rangle - \langle [W,Z],[X,Y]\rangle\\ &= 0\end{aligned}$$

Hence dB vanishes on all left-invariant vector fields, and $(dB)|_g = 0$ for all $g \in G$. So $dB = 0$ and B is a closed 3-form.

When $G = SU(2)$ so that $\mathfrak{g} = su(2) = \left\{ \begin{pmatrix} ai & b+ci \\ -b+ci & -ai \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$,

to find the bi-invariant metric we need ad to be skew-adjoint on $\mathfrak{g}$, i.e:

$$\left\langle \left[\begin{pmatrix} a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & -a_1 i \end{pmatrix}, \begin{pmatrix} b_1 i & b_2 + b_3 i \\ -b_2 + b_3 i & -b_1 i \end{pmatrix} \right], \begin{pmatrix} c_1 i & c_2 + c_3 i \\ -c_2 + c_3 i & -c_1 i \end{pmatrix} \right\rangle =$$

$$= \left\langle \begin{pmatrix} a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & -a_1 i \end{pmatrix}, \left[\begin{pmatrix} b_1 i & b_2 + b_3 i \\ -b_2 + b_3 i & -b_1 i \end{pmatrix}, \begin{pmatrix} c_1 i & c_2 + c_3 i \\ -c_2 + c_3 i & -c_1 i \end{pmatrix} \right] \right\rangle$$

$$2 \left\langle \begin{pmatrix} (a_2 b_3 - a_3 b_2) i & (a_3 b_1 - a_1 b_3) + (a_1 b_2 - a_2 b_1) i \\ -(a_3 b_1 - a_1 b_3) + (a_1 b_2 - a_2 b_1) i & -(a_2 b_3 - a_3 b_2) i \end{pmatrix}, \begin{pmatrix} c_1 i & c_2 + c_3 i \\ -c_2 + c_3 i & -c_1 i \end{pmatrix} \right\rangle =$$

$$= 2 \left\langle \begin{pmatrix} a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & -a_1 i \end{pmatrix}, \begin{pmatrix} (b_2 c_3 - b_3 c_2) i & (b_3 c_1 - b_1 c_3) + (b_1 c_2 - b_2 c_1) i \\ -(b_3 c_1 - b_1 c_3) + (b_1 c_2 - b_2 c_1) i & -(b_2 c_3 - b_3 c_2) i \end{pmatrix} \right\rangle$$

Let's try $\left\langle \begin{pmatrix} a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & -a_1 i \end{pmatrix}, \begin{pmatrix} b_1 i & b_2 + b_3 i \\ -b_2 + b_3 i & -b_1 i \end{pmatrix} \right\rangle = \frac{1}{2}(a_1 b_1 + a_2 b_2 + a_3 b_3)$:

$$c_1(a_2 b_3 - a_3 b_2) + c_2(a_3 b_1 - a_1 b_3) + c_3(a_1 b_2 - a_2 b_1) = \det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} =$$

$$= \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1)$$

which shows that the left-invariant metric generated by this inner product is bi-invariant, and that B on $SU(2)$ is the form $B(X, Y, Z) = \det(X, Y, Z)$.

5. Let (M, g) be a Riemannian manifold, $\omega \in \Omega^k(M)$, and $X_0, \dots, X_k \in \mathrm{Vect}(M)$.
Let $p \in M$ and let $x^1, \dots, x^n$ be normal coordinates at the point p.
Then for $I = (i_1, \dots, i_m)$, $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_m}$, $\omega = \sum_I \omega_I dx^I$, hat indicating omission,

$$\sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \dots, \hat{X}_i, \dots, X_k) = \sum_{i=0}^{k} \sum_I (-1)^i (\nabla_{X_i}\omega_I dx^I)(X_0, \dots, \hat{X}_i, \dots, X_k) =$$

$$= \sum_{i=0}^{k} \sum_I (-1)^i (X_i(\omega_I) dx^I + \omega_I \nabla_{X_i} dx^I)(X_0, \dots, \hat{X}_i, \dots, X_k) =$$

$$= \sum_{i=0}^{k} \sum_I (-1)^i X_i(\omega_I) dx^I (X_0, \dots, \hat{X}_i, \dots, X_k)$$

at p, since at p in normal coordinates $\nabla_{\partial_{x^i}} \partial_{x^j} = 0$ for all i, j, so $\nabla_{X_i} dx^I = 0$ (since $\nabla_{X_i} dx^\ell(\partial_{x_j}) = -dx^\ell(\nabla_{X_i}\partial_{x_j}) = 0$) for all i, j, ℓ).

Now to the left hand side: We evaluate $d\omega$ on the X_i:

$$d\omega(X_0, \dots, X_k) = \sum_I (d\omega_I \wedge dx^I)(X_0, \dots, X_k) = \sum_I \det \begin{pmatrix} d\omega_I(X_0) & \cdots & d\omega_I(X_k) \\ dx^{i_1}(X_0) & \cdots & dx^{i_1}(X_k) \\ \vdots & \ddots & \vdots \\ dx^{i_k}(X_0) & \cdots & dx^{i_k}(X_k) \end{pmatrix} =$$

$$= \sum_I \sum_{j=0}^{k} (-1)^j d\omega_I(X_j) \det \begin{pmatrix} dx^{i_1}(X_0) & \cdots & \widehat{dx^{i_1}(X_j)} & \cdots & dx^{i_1}(X_k) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ dx^{i_k}(X_0) & \cdots & \widehat{dx^{i_k}(X_j)} & \cdots & dx^{i_k}(X_k) \end{pmatrix} =$$

$$= \sum_I \sum_{j=0}^{k} (-1)^j X_j(\omega_I) dx^I (X_0, \dots, \hat{X}_j, \dots, X_k)$$

which is the same as what we found for the right hand side. So we have shown

$$d\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \dots, \hat{X}_i, \dots, X_k)$$