MATH 214 Homework 2

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1 Problem 3-6

Define a function $w : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, $(z_1, z_2) \mapsto \mathrm{Re}[z_1^* z_2]$. Then, in n-dimensional complex vector space $\mathbb{C}^n$, the function $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$, $(u := (u_1, \cdots, u_n), v := (v_1, \cdots, v_n)) \mapsto \sum_{i=1}^n w(u_i, v_i)$ is the same as the inner product on $\mathbb{R}^{2n}$ under the identification $\mathbb{C}^n \leftrightarrow \mathbb{R}^{2n}$. The norm is induced as $\|z\| = \sqrt{(z, z)} = \sqrt{\sum_{i=1}^n |z_i|^2}$ as usual. Then the unit sphere in $\mathbb{C}^2$ is defined by $\mathbb{S}^3 := \{z = (z_1, z_2) \in \mathbb{C}^2 : \|z\|^2 = |z_1|^2 + |z_2|^2 = 1\}$, which is the same as that under the identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. With the natural inclusion $\iota : \mathbb{S}^3 \hookrightarrow \mathbb{C}^2$, I will compute the velocity vector of the curve $\gamma_z : \mathbb{R} \to \mathbb{S}^3$, $t \mapsto (e^{it} z_1, e^{it} z_2)$ under the standard coordinate system in $\mathbb{C}^2$. First, it is obvious that γ_z is a smooth map between $\mathbb{R}$ and $\mathbb{S}^3$ because it multiplies $\gamma_z(0) = (z_1, z_2) \in \mathbb{S}^3$ by a unit-modulus number e^{it} which keeps the point on $\mathbb{S}^3$. By definition, the velocity vector is $\gamma_z'(t_0) := \mathrm{d}\gamma_z(\partial_t|_{t_0}) = ie^{it_0}(z_1 \partial_{z_1} + z_2 \partial_{z_2}) \leftrightarrow (ie^{it} z_1, ie^{it} z_2)$. It can be verified to be a tangent vector because $\langle \gamma_z(t_0), \gamma_z'(t_0) \rangle = 0$. Since $\|\gamma_z'(t_0)\| = \sqrt{|z_1|^2 + |z_2|^2} = 1$, $\forall t_0$, the velocity vector of this smooth curve is never zero.

2 Problem 3-7

For any $p \in M$, let $\Phi : \mathcal{D}_p \to T_pM$ be the natural correspondence defined by $(\Phi v)f = v[f]_p$ where $[f]_p$ is any germ at point p. To prove it is an isomorphism, we need to prove that Φ is bijective.

Suppose $v \in \mathcal{D}_p$. Though the germ f is locally defined as a real-valued function on a neighborhood of point p, Φ is well-defined to map into T_pM by the following reasoning. Given a germ $(f, U) \in C_p^\infty(M)$ where $f : U \to \mathbb{R}$ and $U \subset M$ is a neighborhood of p, let $\tilde{f} : M \to \mathbb{R}$ be a globally defined smooth function that agrees with f on U such that $(\Phi v)\tilde{f} = vf$. Meanwhile, for other globally defined functions, say $\tilde{g}$, which agree with f on U, Proposition 3.8 shows that $(\Phi v)\tilde{f} = (\Phi v)\tilde{g}$. Thus, Φv is independent with the choice of $\tilde{f}$ which means $\Phi : \mathcal{D}_p \to T_pM$ is well defined. Without ambiguity, I just denote $(\Phi v)\tilde{f} = (\Phi v)f$.

First prove the injectivity. Suppose two tangent vectors $\Phi v = \Phi u \in T_pM$. Then for any germ f at p, $(v - u)[f]_p = v[f]_p - u[f]_p = (\Phi v)f - (\Phi u)f = 0$. Because it holds for any real-valued smooth function defined on a neighborhood of p, it means the derivation $v - u = 0$. Thus, $v = u$.

Then prove the surjectivity. Let $\widetilde{v} \in T_pM$ be an arbitary tangent vector and (f, U) be any germ at p. Then, define the derivation on $C_p^\infty(M)$ by $v[f]_p = \widetilde{v}\tilde{f}$ where $\tilde{f}|_U \equiv f$. Because any other $\tilde{g} \in C^\infty(M)$ such that $\tilde{g}|_U \equiv f$ has the same value $\widetilde{v}\tilde{f} = \widetilde{v}\tilde{g}$ by Proposition 3.8. It is evident that the definition of derivation is independent with the choice of $\tilde{f}$. Let this map be $\Phi : v \mapsto \widetilde{v}$. Then,

Φ is surjective.

3 Problem 3-8

Let $\Psi : \mathcal{V}_p \to T_pM$, $[\gamma] \mapsto \Psi[\gamma] = \gamma'(0)$. Let γ_1, γ_2 are two smooth curves on M and $\gamma_1 \sim \gamma_2$. Then for any smooth real-valued function f, $\gamma_1'(0)(f) = \mathrm{d}f(\gamma_1'(0)) = (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) = \mathrm{d}f(\gamma_2'(0)) = \gamma_2'(0)(f)$. Since $\gamma_1 \sim \gamma_2$, by definition, $\gamma_1'(0)(f) = \gamma_2'(0)(f)$ which means $\Psi[\gamma_1] = \Psi[\gamma_2]$. Thus, the map Ψ is well defined.

To prove it is injective, let $\gamma_1'(0) = \gamma_2'(0)$. Then by the argument above, $\gamma_1 \sim \gamma_2$. Thus, Ψ is injective.

To prove it is surjective, let $v \in T_pM$ be any tangent vector. Under the local coordinate, suppose $v = v^j \partial_j$. Then, define the curve $\gamma : (-\epsilon, \epsilon) \to M$ as $\gamma(t) = p + (v^1 t, v^2 t, \cdots, v^n t)$ for small $|t| < \epsilon$. Then, $\gamma(0) = p$, $\gamma'(0) = v$. Therefore, Ψ is surjective.

4 Problem 4-4

Let $(z_1, z_2) = (e^{2\pi i i t_1}, e^{2\pi i t_2}) \in \mathbb{T}^2$ be any point on the torus. Then, $|\gamma(t_1 + n) - (z_1, z_2)| = |e^{2\pi i \alpha(t_1+n)} - e^{2\pi i \alpha t_2}| \leq |2\pi[\alpha(t_1 - t_2 + n) - m]| = 2\pi\,|(\alpha n - m) - (t_2 - \alpha t_1)|$. Because $\alpha\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$ when α is irrational, we can find two integers n, m such that $|(\alpha n - m) - (t_2 - \alpha t_1)| < \epsilon$ for given t_1, t_2 and $\forall \epsilon$. Thus, for any point on the torus and ϵ, there exists an integer n such that $|\gamma(t_1 + n) - (z_1, z_2)| < 2\pi\epsilon$. Therefore, γ is dense in $\mathbb{T}^2$.

5 Problem 4-6

Suppose there is a smooth submersion $F : M \to \mathbb{R}^k$ for a given $k > 0$. Then by Proposition 4.28, it is an open map. Meanwhile, because M is compact and $\mathbb{R}^k$ is Hausdorff, F is a close map. Therefore, because nonempty manifold M is both open and close in its topology, $F(M)$ is both open and close in $\mathbb{R}^k$. Because $\mathbb{R}^k$ is connected, it implies that $F(M) = \mathbb{R}^k$. Moreover, since F is continuous and M is compact, $F(M) = \mathbb{R}^k$ is compact. That contradicts that $\mathbb{R}^k$ is not compact. Thus, for any $k > 0$ and a nonempty compact manifold M, there doesn't exist a smooth submersion $F : M \to \mathbb{R}^k$.