MATH 214 Homework 3

Yulong Dong

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1 Problem 5-1

The differential of the map is

$$\mathrm{d}\Phi(x, y, s, t) = (2x\mathrm{d}x + \mathrm{d}y, 2x\mathrm{d}x + (2y + 1)\mathrm{d}y + 2s\mathrm{d}s + 2t\mathrm{d}t) \tag{1}$$

The preimage of the point $(0, 1)$ is determined by the following smooth equations

$$\begin{cases} x^2 + y = 0 \\ x^2 + y^2 + s^2 + t^2 + y = 1 \end{cases} \Leftrightarrow \begin{cases} x^2 + y = 0 \\ y^2 + s^2 + t^2 = 1 \end{cases} \tag{2}$$

Thee appearence of $\mathrm{d}y$ makes $\mathrm{d}\Phi \neq 0$. It remains to show that two components of $\mathrm{d}\Phi$ are linear independent. Suppose $(\mathrm{d}\Phi)^1 \propto (\mathrm{d}\Phi)^2$, then $y = s = t = 0$ or $x = s = t = 0$. Both cases are forbidden by the equations of $\Phi^{-1}(0, 1)$. That means, $\mathrm{d}\Phi$ at any point in $\Phi^{-1}(0, 1)$ is surjective. Then, $(0, 1)$ is a regular value of Φ.

Let $S = \Phi^{-1}(0, 1)$. First define a function $\phi_1 : \mathbb{R}^4 \to \mathbb{R}^4,\ (x, y, s, t) \mapsto (x, x^2 + y, s, t)$. It is evident that ϕ_1 is bijective and smooth. Its inverse $\phi_1^{-1} : (x, y, s, t) \mapsto (x, y - x^2, s, t)$ is also smooth. Thus, $\phi_1 : \mathbb{R}^4 \to \mathbb{R}^4$ is a diffeomorphism. Then, $S \overset{\phi_1}{\simeq} \phi_1(S) = \{(x, 0, y, z) : x^4 + y^2 + z^2 = 1\}$. It remains to show that $S' = \{(x, y, z) : x^4 + y^2 + z^2 = 1\}$ is diffeomorphic to $\mathbb{S}^2$. Define $\phi_2 : \mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (x, y\sqrt{1 + x^2}, z\sqrt{1 + x^2})$. Because $x^4 + (1 + x^2)y^2 + (1 + x^2)z^2 = x^2(x^2 + y^2 + z^2) + y^2 + z^2 = 1 \Leftrightarrow x^2 + y^2 + z^2 = 1$. It can be concluded that ϕ_2 is bijective and smooth. Its inverse $\phi_2^{-1} : (x, y, z) \mapsto (x, \frac{x}{\sqrt{1+x^2}}, \frac{z}{\sqrt{1+x^2}})$ is smooth. Thus, $\phi_2 : \mathbb{S}^2 \to S'$ is a diffeomorphism. Therefore, $S \simeq \mathbb{S}^2$.

2 Problem 5-6

Define a function $f : T\mathbb{R}^n \to \mathbb{R},\ (p, v) \mapsto |v|^2$. Its differential is $\mathrm{d}f = 2\sum_{i=1}^n v^i \mathrm{d}v^i$ which is nonvanishing when v^i's are not all zero. Thus, $f^{-1}(1)$ is the set of regular point. Because the unit tangent bundle is a regular level set of f, *i.e.*, $UM = f^{-1}(1)$, it follows that UM is an embedded submanifold of dimension $2m - 1$.

3 Problem 5-15

Take the figure-eight curve for example. Take two immersion maps $(-\pi, \pi) \to \mathbb{R}^2$ as

$$\beta_1(t) = (\sin(2t), \sin t), \quad \beta_2(t) = (-\sin(2t), \sin t) \tag{3}$$

Both two maps are smooth immersion and map $(-\pi, \pi)$ onto the same subset in $\mathbb{R}^2$, *i.e.*, figure-eight curve and denote it as S, in different ways. One can find the difference through $\beta_1'(0) = (2,1), \beta_2'(0) = (-2,1)$ and $\beta_1(\pi - \epsilon) = (-,+), \beta_2(\pi - \epsilon) = (+,+)$, *etc.* It can be visualized by the following figure.

By Proposition 5.18, we are able to define the topology and smooth structure *w.r.t.* β_1, β_2 such that S is a smooth submanifold in $\mathbb{R}^2$ and $(-\pi, \pi) \simeq S$. For each $\beta_j, j = 1, 2$, define $U \subset S$ to be open if $\beta_j^{-1}(U) \subset (-\pi, \pi)$ is open. It is evident that the topologies indeced by β_j's are different. The smooth structure is picked as $(\beta_j(U), \phi \circ \beta_j^{-1})$ for a chart (U, ϕ) in $(-\pi, \pi)$.

4 Problem 5-18

(1)

"$\Rightarrow$":

Let S be an k-embedded submanifold in M^m and $f : S \to N$ is a smooth function on S. Then, each point $p \in S$, one can find a neighborhood $p \in U \subset M$ and a k-slice coordinate $\phi : U \to \mathbb{R}^m$ such that $\phi(U \cap S) \subset \{(x^1, \cdots, x^k, x^{k+1}, \cdot, x^m) : x^{k+1} = \cdots = x^m = 0\}$. Due to the second contable axiom, S can be covered by neighborhoods of finitely many points on it, *i.e.*, $\{U_i, \phi_i\}$ where $U_i \ni p_i$ and ϕ_i is the corresponding k-slice chart. Let $\{\rho_i\}$ be a partition of unity that each ρ_i subordinates to U_i. Define a smooth function f_i on each U_i as $f_i \circ \phi_i^{-1}(x^1, \cdots, x^m) \equiv f \circ \phi_i^{-1}(x^1, \cdots, x^k, 0 \cdots, 0)$. That is function f_i in the slice chart is independent with coordinates $x^{k+1}, \cdots, x^m$. Then, $f_i|_{S \cap U_i} = f|_{S \cap U_i}$. Use the POU to glue them up and define $\widetilde{f} = \sum_i f_i \rho_i$. Since each p has a neighborhood intersecting finitely many U_i's in POU, the summation is finite for each point. That means $\widetilde{f} : U \to N$ is smooth and $\widetilde{f}|_S = f$ where $U = \cup_i U_i$ is a neighborhood of S in M.

"$\Leftarrow$":

Suppose S is an immersed submanifold in M, then by the fact that immersed submanifold is locally embedded, each $p \in S$, there is a neighborhood U of p in S that is embedded. Let $f \in C^\infty(S)$ such that $\mathrm{supp}(f) \subset U$, $f(p) = 1$. It can be smoothly extended to $\widetilde{f}$ on a neighborhood V of S in M. By continuity, $W := \widetilde{f}^{-1}((0, \infty))$ is open in M. Thus, $W \cap S$ is open in S. Since $f(p) = \widetilde{f}(p) = 1 > 0$, $p \in W \cap S$. Because U is embedded, there exists a local chart U_p centered at p such that $U \cap U_p$ is k-slice. Due to the support of f, $U_p \cap W \cap S \subset U_p \cap W \cap U$, which means each point p has a local k-slice chart $U_p \cap W \cap S$. Thus, S is an embedded submaniifold.

(2)

"$\Rightarrow$":

The proof is similar to that in (1). The only difference is that we can now add the open subset $M \backslash S$ into the cover. By assigning another POU ρ_0 supported in $M \backslash S$, the extended function is defined as $\widetilde{f} = \rho_0 + \sum_i f_i \rho_i$. This function is then defined globally on M. Moreover, since $\text{supp}(\rho_0) \subset M \backslash S$, $\widetilde{f}|_S = f|_S$ as argued in (1).

"$\Leftarrow$":

Because $f \in C^\infty(S)$ has a smooth extension to M, by the lemma in (1), it can be concluded that S is an embedded submanifold in M. Let $\iota : S \hookrightarrow M$ be the inclusion map that makes it into an embedded submanifold. It remains to show that S is eembedded properly, *i.e.*, ι is proper. Suppose there exist a compact subset $K \subset M$ such that $\iota^{-1}(K)$ is noncompact in S. Then, there exists a sequence of points $\{p_i\}$ without limit point. That means for each point in $\iota^{-1}(K)$, any neighborhood of it contains finitely many points in the sequence. Moreover, we can find neighborhood of each point in the sequence $\{U_i \ni p_i\}$ such that they are disjoint. Assign a function ϕ_i on each U_i such that $\phi_i(p_i) = i$ and $\text{supp}(\phi_i) \subset U_i$. Then the function $\phi = \sum_i \phi_i$ is smooth because finite terms involve when evaluating at each point. By assumption, it can be extended globally to $\widetilde{\phi} \in C^\infty(M)$ such that $\widetilde{\phi}|_S = \phi$, namely, $\widetilde{\phi} \circ \iota = \phi$. By construction, $\widetilde{\phi}(\iota(p_i)) \to \infty$. However, $\iota(p_i) \in K$ which is compact in M. $\widetilde{\phi}$ is smooth which means the image $\widetilde{\phi}(K)$ is also compact. That is a contradiction. Thus, ι is a proper embedding.

5 Outline of the proof of Whitney Embedding Theorem

The proof neeeds the following lemmas that is proved in the compact case.

Lemma 5.1. Any smooth manifold that can be covered by finitely many coordinate charts has an injective immersion into $\mathbb{R}^K$ for sufficiently large K.

Lemma 5.2. If a smooth m-manifold has an injective immersion into $\mathbb{R}^K$ and $K > 2m + 1$, then it has an injective immersion into $\mathbb{R}^{K-1}$.

<u>First step.</u>

Lemma 5.3. Any non-compact smooth manifold has an injective immersion into $\mathbb{R}^K$ for sufficiently large K.

Proof. With Proposition 2.28, we are able to find smooth exhaustion functionf f such that $f^{-1}((-\infty, c])$ is compact in M. By the continuity of f, we define $M_i = f([i, i+1])$, $i \in \mathbb{Z}$ which are compact subsets in M. The compactness impliees that M_i can be covered by finitely many open sets $\{U_j := 1, \cdots, k_i\}$. By slightly enlarging M_i to be an open subset, we define $N_i = \left(\cup_{j=1}^{k_i} U_j\right) \cap f^{-1}((i - \epsilon, i + 1 + \epsilon))$ where $\epsilon \in (0, 1)$. It is evident that $N_i \cap N_j = \emptyset$ if $|i - j| \geq 2$. Because N_i, an open submanifold in M, can be covered by finite open subsets, by Lemma 5.1 and Lemma 5.2, we can find injective immersion $\phi_i : N_i \to \mathbb{R}^{2m+1}$.

We now assign a bump function ρ_i to each N_i such that $\rho_i = 1$ in an open neighborhood of M_i and $\text{supp}(\rho_i) \subset N_i$. Define a function as

$$\Phi : M \to \mathbb{R}^{4m+3},\ p \mapsto \left(\sum_{i \in 2\mathbb{Z}} \rho_i(p)\phi_i(p),\ \sum_{i \in 2\mathbb{Z}+1} \rho_i(p)\phi_i(p), f(p) \right)$$

It is important that in two summations, **at most one term in each is nonzero** because $N_i \cap N_j = \emptyset$ if $|i - j| \geq 2$. Then, it is smooth. We need to show that Φ is an injective immersion.

1. injectivity. $\Phi(p_1) = \Phi(p_2) \Rightarrow \exists i \in \mathbb{Z}$ s.t. $f(p_1) = f(p_2) \in [i, i+1] \Rightarrow p_1, p_2 \in M_i \subset N_i \Rightarrow \phi_i(p_1) = \phi_i(p_2) \Rightarrow p_1 = p_2$. Here, the properties that ϕ_i is injective and that of the summation are used.
2. immersion. WLOG, let $\phi_i(p) \neq 0$ and $i \in 2\mathbb{Z}$. Then, $\mathrm{d}\Phi(p) = (\mathrm{d}\phi_i(p), *, *)$. The fact that ϕ_i is an immersion implies that Φ is an immersion.

□

Second step. By Lemma 5.2 and Lemma 5.3, it can be concluded that there exists an injective immersion $\Phi : M \to \mathbb{R}^{2m+1}$ for any non-compact manifold M.

Third step. We need another lamma to prove Whitney embedding theorem.

Lemma 5.4. A proper injective immersion is an embedding.

Theorem 5.5 (Whitney Embedding Theorem)**.** Any smooth non-compact m-manifold can be embedded into $\mathbb{R}^{2m+1}$.

Proof. Let $\Phi : M \to \mathbb{R}^{2m+1}$ is an injective immersion. Composite Φ with a diffeomorphism onto unit ball $\mathbb{R}^{2m+1} \to \mathbb{B}_{2m+1},\ x \mapsto \frac{x}{1+|x|^2}$. For simplicity, denote the composite map again by Φ. Let $v \in \mathbb{S}^{2m+1}$ and $\pi_v : \mathbb{R}^{2m+2} \to P_v = \{u \in \mathbb{R}^{2m+2} : u \cdot v = 0\} \simeq \mathbb{R}^{2m+1},\ x \mapsto x - (x \cdot v)v$ be the proection onto the normal hyperplane of given direction $v = (v', v^{2m+2})$. Define a function as

$$\widetilde{\Phi} : M \to \mathbb{R}^{2m+2},\ p \mapsto (\Phi(p), f(p)), \quad \Psi = \pi_v \circ \widetilde{\Phi},\ p \mapsto \left(*, f(p)\left(1 - (v^{2m+2})^2\right) - \left(\Phi(p) \cdot v'\right) v^{2m+2}\right)$$

Because Φ is an injective immersion, $\widetilde{\Phi}$ is an injective immersion. We now need to reduce the dimension of the target space by the projection π_v. By a similar argument as in the compact case, Ψ is an injective immersion almost everywhere in $\mathbb{S}^{2m+1}$. Then, we need to show that Ψ is proper.

Let $K \subset \{x \in \mathbb{R}^{2m+2} : |x_{2m+2}| < A\}$ be a compact subset for some $A > 0$. Choose v such that $\left|v^{2m+2}\right| < 1$. Because $|\Phi(p)|, |v'|, \left|v^{2m+2}\right| < 1$, it follows that

$$|f(p)| < \frac{A+1}{|(1 - (v^{2m+2})^2)|}. \tag{4}$$

Then, $\Psi^{-1}(K) \subset f^{-1}([-\frac{A+1}{|(1-(v^{2m+2})^2)|}, \frac{A+1}{|(1-(v^{2m+2})^2)|}])$ which is a subset of a compact subset of M. The continuity of Ψ implies that $\Psi^{-1}(K)$ is closed. Thus, $\Psi^{-1}(K)$ is compact. By definition, Ψ is proper. □