MATH 214 Homework 6

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1 Corollary 9.39

Rhe following theorem will be used in proving this corollary.

Theorem 1.1 (Theorem 9.38 in Lee's book)**.** If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$, then $\mathcal{L}_V W = [V, W]$.

Suppose M is a smooth manifold with or without boundary, and $V, W, X \in \mathfrak{X}(M)$.

1.1

$\mathcal{L}_V W = -\mathcal{L}_W V$.

Proof. By Theorem 1.1, left hand side $= [V, W] = -[W, V] = -\mathcal{L}_W V =$ right hand side, where the property of Lie bracket is used to interchange two vector fields in Lie bracket. □

1.2

$\mathcal{L}_V [W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.

Proof. It is basically Jacobi identity, left hand side $= [V, [W, X]] = [[V, W], X] + [W, [V, X]] =$ right hand side. □

1.3

$\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W - \mathcal{L}_W \mathcal{L}_V X$.

Proof. It is again Jacobi identity, left hand side $= [[V, W], X] = [V, [W, X]] - [W, [V, X]] =$ right hand side. □

1.4

If $g \in C^\infty(M)$, then $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$.

Proof. Use Proposition 8.28 (d) in Lee's book, $[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X, \forall f, g \in C^\infty(M), X, Y \in \mathfrak{X}(M)$. Specifically take $f \equiv 1$ be a constant function and note that $Y(f) = 0$ since smooth vector field can be identified as derivation on smooth functions, it becomes $[X, gY] = g[X,Y] + X(g)Y$. Rewrite them in terms of Lie derivative by Theorem 1.1, $\mathcal{L}_X(gY) = X(g)Y + g\mathcal{L}_X Y$. □

1.5

If $F : M \to N$ is a diffeomorphism, then $F_*(\mathcal{L}_V X) = \mathcal{L}_{F_* V} F_* X$.

Proof. Use Corollary 8.31 in Lee's book, we have $F_*([V,X]) = [F_*V, F_*X]$ when F is a diffeomorphism. Thus, by Theorem 1.1, left hand side $= F_*([V,X]) = [F_*V, F_*X] = \mathcal{L}_{F_*V}(F_*X) =$ right hand side. □

2 Problem 9-8

$S \subset M$ is an embedded submanifold and $V \in \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Let $\theta : \mathcal{D} \to M$ be the flow of V. Because S is a compact embedded submanifold, V is complete on S due to Corollary 9.17, which means $\pi_2(\mathcal{D}) \supset S$. Let $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D} = \mathcal{D}' \times S$ where $\mathcal{D}' \subset \mathbb{R}$ and $\Phi = \theta|_{\mathcal{O}}$. Then, use Theorem 9.20 in Lee's book, $\Phi : \mathcal{O} \to M$ is a smooth submersion and there exists a smooth positive function $\delta : S \to \mathbb{R}$ such that $\Phi|_{\mathcal{O}_\delta}$ is injective where $\mathcal{O}_\delta = \{(t,p) \in \mathcal{O} : |t| < \delta(p)\}$. Because S is compact ad δ is cotiuous, the image $\delta(S) \in \mathbb{R}_+$ is compact, say $\delta(S) = [\alpha, \beta]$ where $0 < \alpha < \beta$. Then, let $\epsilon = \frac{\alpha}{2}$, $[-\epsilon, \epsilon] \times S = \overline{\mathcal{O}_\epsilon} \subset \mathcal{O}_\delta$ and $\Phi(\overline{\mathcal{O}_\epsilon})$ is an immersed submanifold of M. Because $\overline{\mathcal{O}_\epsilon}$ is compact, by Proposition 5.21, $\Phi(\overline{\mathcal{O}_\epsilon})$ is embedded submanifold in M. Therefore, there exists $\epsilon > 0$ such that $\mathcal{O}_\epsilon = (-\epsilon, \epsilon) \times S$ and $\Phi : \mathcal{O}_\epsilon \to M$ is a smooth embedding.

3 Problem 14-5

First prove that $\alpha^i \in \text{span}\{\omega^j : j = 1, \cdots, k\}$. For any $i = 1, \cdots, k$, $\alpha^i \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^k = (-1)^{i-1}\omega^1 \wedge \cdots \wedge (\alpha^i \wedge \omega^i) \wedge \cdots \wedge \omega^k = (-1)^i \sum_{j=1, j\neq i}^k \omega^1 \wedge \cdots \wedge (\alpha^j \wedge \omega^j) \wedge \cdots \wedge \omega^k = 0$. Thus, $\alpha^i \in \text{span}\{\omega^j : j = 1, \cdots, k\}$. Because $\text{span}\{\omega^j : j = 1, \cdots, k\}$ is a smooth subbundle and α^i's are smooth 1-form on U, by Proposition 10.22, the component functions of α^i in terms of the local frame $\{\omega^j : j = 1, \cdots, k\}$ is smooth. Thus, each α^i can be written as a linear combination of $\omega^1, \cdots, \omega^k$ with smooth coefficients.

4 Problem 14-6

4.1

$$\begin{aligned} dx &= \frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi \\ &= \sin\varphi \cos\theta d\rho - \rho \sin\varphi \sin\theta d\theta + \rho \cos\varphi \cos\theta d\varphi \\ dy &= \frac{\partial y}{\partial \rho} d\rho + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi \\ &= \sin\varphi \sin\theta d\rho + \rho \sin\varphi \cos\theta d\theta + \rho \cos\varphi \sin\theta d\varphi \\ dz &= \frac{\partial z}{\partial \rho} d\rho + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi \\ &= \cos\varphi d\rho - \rho \sin\varphi d\varphi \end{aligned} \tag{1}$$

Then,

$$\begin{aligned} dy \wedge dz &= -\rho \sin\varphi \cos\varphi \cos\theta d\rho \wedge d\theta - \rho^2 \sin^2\varphi \cos\theta d\theta \wedge d\varphi + \rho \sin\theta d\varphi \wedge d\rho \\ dz \wedge dx &= -\rho \sin\varphi \cos\varphi \sin\theta d\rho \wedge d\theta - \rho^2 \sin^2\varphi \sin\theta d\theta \wedge d\varphi - \rho \cos\theta d\varphi \wedge d\rho \\ dx \wedge dy &= \rho \sin^2\varphi d\rho \wedge d\theta - \rho^2 \sin\varphi \cos\varphi d\theta \wedge d\varphi \end{aligned} \tag{2}$$

Thus,

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy = -\rho^3 \sin\varphi d\theta \wedge d\varphi \tag{3}$$

4.2

In Cartesian coordinate,

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy = 3 dx \wedge dy \wedge dz \tag{4}$$

In spherical coordinate,

$$d\omega = -3\rho^2 \sin\varphi d\rho \wedge d\theta \wedge d\varphi \tag{5}$$

Use Equation (1), Equation (2), it can be shown that

$$dx \wedge dy \wedge dz = \left(\rho \sin^2\varphi d\rho \wedge d\theta - \rho^2 \sin\varphi \cos\varphi d\theta \wedge d\varphi\right) \wedge (\cos\varphi d\rho - \rho \sin\varphi d\varphi) = -\rho^2 \sin\varphi d\rho \wedge d\theta \wedge d\varphi \tag{6}$$

Thus, both expressions represent the same 3-form $d\omega$.

4.3

Let the inclusion map be $\iota : \mathbb{S}^2 \to \mathbb{R}^3$, $(\varphi, \theta) \mapsto (x, y, z) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$. The coordinates (φ, θ) is well defined on the open subset $(0, \pi) \times (0, 2\pi)$. Then, the pullback of ω to $\mathbb{S}^2$ is

$$\iota^* \omega = \sin\varphi d\varphi \wedge d\theta \tag{7}$$

4.4

For point $p \in \mathbb{S}^2$ in the spherical coordinate chart (φ, θ), it is evident that $\iota^*\omega|_p \neq 0$ because $\sin\varphi \neq 0$. For the north and south pole, *i.e.*, $\varphi = 0, \pi$ which can not be described by spherical coordinate system, consider the 2-form in $\mathbb{R}^3$ which is $\omega|_{\text{pole}} = \pm 1 dx \wedge dy$. Because the tangent space of $\mathbb{S}^2$ at both poles as an embedded submanifold in $\mathbb{R}^3$ is parallel to $x-y$ plane, the pullback $\iota^*\omega$ is nonzero at those two poles. Thus, it can be concluded that $\iota^*\omega$ is nowhere zero.

5 Sketch of the proof of Theorem 9.38

Theorem 5.1 (Theorem 9.38 in Lee's book)**.** If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$, then $\mathcal{L}_V W = [V, W]$.

Proof. Let $\mathcal{R}(V) \subset M$ be the regular points of V. By continuity, it is open. Consider points in M in different cases.

1. $p \in \mathcal{R}(V)$. By Theorem 9.22, we can choose canonical coordinate chart (u^i) near p such that $V = \frac{\partial}{\partial u^1}$ which means the flow in the chart is $\theta_t(u) = (u^1 + t, u^2, \cdots, u^n)$. Thus, $d(\theta_{-t})_{\theta_t(x)}$ is identity at every point for fixed t. Then, by the definition of Lie derivative, $(\mathcal{L}_V W)_u = \sum_j \frac{\partial W^j}{\partial u^1}(u^1, \cdots, u^n) \frac{\partial}{\partial u^j}|_u$ which is the same as the Lie bracket $[V, W]_u$.
2. $p \in \operatorname{supp}(V) = \overline{\mathcal{R}(V)}$. By continuity.
3. $p \in M \backslash \operatorname{supp}(V)$. $V = 0$ on a neighborhood of p implies that θ_t is identity in that neighborhood for all t. Use the definition of Lie derivative, $(\mathcal{L}_V W)_p = 0$.

□