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Math 214, Homework 7.

Ex. 14.28. Recall the grad operator $\nabla : C^\infty(\mathbb{R}^3) \to \mathfrak{X}(\mathbb{R}^3)$, the curl operator $\nabla\times : \mathfrak{X}(\mathbb{R}^3) \to \mathfrak{X}(\mathbb{R}^3)$, and the divergence operator $\nabla\cdot : \mathfrak{X}(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$. We also have the index-lowering isomorphism $\flat : \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ which sends $X^i \frac{\partial}{\partial x^i} \mapsto X^i \delta_{ij} dx^j$, the isomorphism $\beta : \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ sending $X \mapsto X \lrcorner (dx \wedge dy \wedge dz)$ and the isomorphism $* : C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ sending $f \mapsto f\, dx \wedge dy \wedge dz$.
We want to show that the following diagram commutes:

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\nabla} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla\times} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla\cdot} & C^\infty(\mathbb{R}^3) \\
\big\downarrow \mathrm{Id} & & \big\downarrow \flat & & \big\downarrow \beta & & \big\downarrow * \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$

We'll show the first and last squares for $\mathbb{R}^n$. To begin with:
Let $f \in C^\infty(\mathbb{R}^n)$. Then $\flat(\nabla f) = \flat\left(\sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}\right) = \frac{\partial f}{\partial x^i}\delta^i_j dx^j = \frac{\partial f}{\partial x^j} dx^j = df$
So the first square commutes for $\mathbb{R}^n$, and in particular for $\mathbb{R}^3$.

For the last square, let $X = X^j \frac{\partial}{\partial x^j} \in \mathfrak{X}(\mathbb{R}^n)$.
Then $*(\nabla \cdot X) = *(\sum_j \frac{\partial X^j}{\partial x^j}) = \left(\sum_j \frac{\partial X^j}{\partial x^j}\right) \bigwedge_k dx^k$, while
$d(\beta(X)) = d\left(X \lrcorner \bigwedge_k dx^k\right) = d\left(\sum_j (-1)^{j+1} X^j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n\right) =$
$= \sum_j (-1)^{j+1} dX^j \wedge dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n = \left(\sum_j \frac{\partial X^j}{\partial x^j}\right) \bigwedge_k dx^k$.
So the last square commutes for $\mathbb{R}^n$, and in particular for $\mathbb{R}^3$.

Finally, the middle square commutes for $\mathbb{R}^3$. Indeed, for $X \in \mathfrak{X}(\mathbb{R}^3)$ we compute:
$\beta(\nabla \times X) = \beta\left(\left(\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y}\right)\frac{\partial}{\partial z}\right) =$
$= \left(\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z}\right) dy \wedge dz - \left(\frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x}\right) dx \wedge dz + \left(\frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y}\right) dx \wedge dy$
while $d(\flat(X)) = d\left(X^1 dx + X^2 dy + X^3 dz\right) =$
$= \frac{\partial X^1}{\partial y} dy \wedge dx + \frac{\partial X^1}{\partial z} dz \wedge dx + \frac{\partial X^2}{\partial x} dx \wedge dy + \frac{\partial X^2}{\partial z} dz \wedge dy + \frac{\partial X^3}{\partial x} dx \wedge dz + \frac{\partial X^3}{\partial y} dy \wedge dz =$
$= \left(\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z}\right) dy \wedge dz - \left(\frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x}\right) dx \wedge dz + \left(\frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y}\right) dx \wedge dy$
So the middle square commutes for $\mathbb{R}^3$, and the whole diagram commutes for $\mathbb{R}^3$.
In particular, since $d^2 = 0$ and the vertical maps are isomorphisms, hence invertible, for any $f \in C^\infty(\mathbb{R}^3)$ we have $\nabla \times (\nabla f) = 0$ and for any $X \in \mathfrak{X}(\mathbb{R}^3)$ we have $\nabla \cdot (\nabla \times X) = 0$.

7-2. Let G be a Lie group.

(a) Let $m : G \times G \to G$ be the multiplication map, and identify $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$. Let (U, φ) be a chart at $e \in G$ with $\varphi(0) = e$, so that $(U \times U, \varphi \times \varphi)$ is a chart at $(e, e) \in G \times G$. Then for $X \in \mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\varphi(m((\varphi \times \varphi)^{-1}(0+\varepsilon X, 0))) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\varphi(m(\varphi^{-1}(\varepsilon X), e)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\varphi(\varphi^{-1}(\varepsilon X)) = X$$

Going back to G, this shows that $dm_{(e,e)}(X, 0) = X$.
Doing the same calculation as above with εX on the right, $dm_{(e,e)}(0, X) = X$.
Then by linearity, $dm_{(e,e)}(X, Y) = X + Y$.

(b) Let $i : G \to G$ be the inversion map. Letting $f : G \to G$ be the identity map, we find that $(m \circ (f \times i))(g) = e$ for any $g \in G$. Then this is a constant map and we find that for $X \in T_eG$, $0 = d(m \circ (f \times i))_e(X) = dm_{(e,e)}(df_eX, di_eX) = X + di_eX$ by the chain rule and the fact that $d(f \times i) = df \times di$. So we've shown $di_eX = -X$.

7-4. (a) For $A \in \mathrm{M}(n, \mathbb{R})$, $\det(I_n + tA)$ is a polynomial in t, where the linear term is only given by products of elements picked from the diagonal, of which $(n - 1)$ must be 1's and the other linear in t, so terms of the form $1 \cdots 1 \cdot tA^j_j \cdot 1 \cdots 1 = tA^j_j$.
So the linear coefficient is $\sum_{j=1}^n A^j_j = \mathrm{tr}(A)$. Then $\frac{d}{dt}\big|_{t=0} \det(I_n + tA) = \mathrm{tr}(A)$.

(b) For $X \in \mathrm{GL}(n, \mathbb{R})$ and $B \in T_X\mathrm{GL}(n, \mathbb{R}) \cong \mathrm{M}(n, \mathbb{R})$, let $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{GL}(n, \mathbb{R})$ be the curve defined by $\gamma(t) = X + tB$. Then $\gamma(0) = X$, $\gamma'(0) = B$, and

$$d(\det)_X(B) = (\det \circ \gamma)'(0) = \frac{d}{dt}\Big|_{t=0} \det(X+tB) = \frac{d}{dt}\Big|_{t=0} \det(X)\det(I_n+tX^{-1}B) =$$

$$= \det(X)\frac{d}{dt}\Big|_{t=0} \det(I_n + tX^{-1}B) = \det(X)\mathrm{tr}(X^{-1}B)$$

which is the desired result.

7-9. Define a map $\cdot : \mathrm{GL}(n+1,\mathbb{R}) \times \mathbb{RP}^n \to \mathbb{RP}^n$ by $A \cdot [x] = [Ax]$.

This is well defined: if $[x] = [y]$, then $x = cy$ for some c and $Ax = cAy$ so $[Ax] = [Ay]$.

This is a left action, because we have $I_n \cdot [x] = [I_n x] = [x]$ and for $A, B \in \mathrm{GL}(n+1,\mathbb{R})$, we have $A \cdot (B \cdot [x]) = A \cdot [Bx] = [A(Bx)] = [(AB)x] = (AB) \cdot [x]$.

The action is transitive because for any nonzero $x, y \in \mathbb{R}^{n+1}$ we may pick bases for $\mathbb{R}^{n+1}$, the first containing x and the second containing y, so that there is an invertible linear map A in $\mathrm{GL}(n+1,\mathbb{R})$ carrying the first basis to the second and x to y.

Then $A \cdot [x] = [Ax] = [y]$. So the action is transitive.

We check that the action is smooth in charts. Let $V_i = \{[x^1, \ldots, x^{n+1}] : x^i \neq 0\} \subseteq \mathbb{RP}^n$. Then we have the charts $\varphi_i : V_i \to \mathbb{R}^n : [x^1, \ldots, x^{n+1}] \mapsto \frac{1}{x^i}(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1})$ which cover $\mathbb{RP}^n$ and to satisfy one of our smoothness characterizations we have that $(\mathrm{GL}(n+1,\mathbb{R}) \times V_j) \cap (\cdot)^{-1}(V_i) = \{(A, [x]) : x^j \neq 0, (Ax)^i \neq 0\}$ is open.

We also have the inclusion $i : \mathrm{GL}(n+1,\mathbb{R}) \hookrightarrow \mathrm{M}(n+1,\mathbb{R}) \cong \mathbb{R}^{(n+1)^2}$ of an open set so that we have reduced to checking the smoothness of
$\varphi_i \circ (\cdot) \circ (i \times \varphi_j)^{-1} : (i \times \varphi_j)\left((\mathrm{GL}(n+1,\mathbb{R}) \times V_j) \cap (\cdot)^{-1}(V_i)\right) \to \varphi_i(V_i)$.
If we notate $x^j = 1$, the formula for this is just $(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^{n+1}) \mapsto$
$\varphi_i(A \cdot [x^1, \ldots, x^{n+1}]) = \varphi_i([A^1_k x^k, \cdots, A^{n+1}_k x^k]) = \frac{1}{A^i_k x^k}\left(A^1_k x^k, \cdots, A^{i-1}_k x^k, A^{i+1}_k x^k, \cdots, A^{n+1}_k x^k\right)$
(using Einstein summation) which is smooth wherever $A^i_k x^k = (Ax)^i \neq 0$.

So we have shown that this is a smooth transitive left action.

7-11. Consider $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$, and the action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ by $z \cdot (w^1, \ldots, w^{n+1}) = (zw^1, \cdots, zw^{n+1})$.

This action is smooth because it is the restriction of the smooth map
$\mathbb{C}^{n+2} \to \mathbb{C}^{n+1} : (z, w^1, \ldots, w^{n+1}) \mapsto (zw^1, \ldots, zw^{n+1})$ to domain and range embedded submanifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$ and $\mathbb{S}^{2n+1}$, which is smooth.

Now, the orbit through $w = (w^1, \ldots, w^{n+1})$ is $\{(e^{i\theta} w^1, \ldots, e^{i\theta} w^{n+1}) : \theta \in [0, 2\pi)\}$,

which is a unit circle in $\mathbb{C}^{n+1}$ because $|w| = 1$.

Any two distinct orbits are disjoint, because if they share a point then the orbit generated by this point contains both orbits. Furthermore, there is such a unit circle orbit through any point in $\mathbb{S}^{2n+1}$.

So the orbits split $\mathbb{S}^{2n+1}$ into a union of disjoint unit circles.