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Math 214, Homework 8.

7-13. Let $n \geq 1$ and $U(n) := \{A : A^*A = I\} \subseteq \mathrm{GL}(n, \mathbb{C})$. $U(n)$ is a group because $I \in U(n)$ and for $A, B \in U(n)$, $(AB)^*(AB) = B^*A^*AB = B^*B = I$.
We show $U(n)$ is a properly embedded submanifold of $\mathrm{GL}(n, \mathbb{C})$ with the constant rank theorem and the equivariant rank theorem on $F : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{M}(n, \mathbb{C})$, $F(A) = A^*A$.

If $\mathrm{GL}(n, \mathbb{C})$ acts on itself by right multiplication and on $\mathrm{M}(n, \mathbb{C})$ by conjugation:
For $A, B \in \mathrm{GL}(n, \mathbb{C})$, $F(AB) = (AB)^*(AB) = B^*(A^*A)B = B^*F(A)B$.
The right action on $\mathrm{GL}(n, \mathbb{C})$ is transitive so F is equivariant and constant rank.
Then $U(n) = F^{-1}(I)$ is a properly embedded submanifold of $\mathrm{GL}(n, \mathbb{C})$.
Thus $U(n)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$.

To find its dimension, we compute the rank of F at I. For $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, $B \in \mathrm{M}(n, \mathbb{C})$, $F(I + tB) = (I + tB)^*(I + tB) = I + t(B^* + B) + \mathcal{O}(t^2)$,
so the range of dF_I is the space of self-adjoint matrices in $\mathrm{M}(n, \mathbb{C})$.
Over $\mathbb{R}$, a basis is given by the matrices $E_{jk} + E_{kj}$, $iE_{jk} - iE_{kj}$ for $j < k$, and E_{jj} (where E_{jk} is the matrix with zeroes and a 1 in position j, k).
There are $2 \times \frac{1}{2}(n-1)n + n = n^2$ basis elements. Since $\{I\}$ is zero dimensional, the codimension of $U(n)$ in $\mathrm{GL}(n, \mathbb{C})$ is n^2 and the dimension of $U(n)$ is $2n^2 - n^2 = n^2$.

8-19. $\mathbb{R}^3$ is a real vector space. We check that $[\cdot, \cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined $[x, y] = x \times y$ is in fact a Lie bracket. We check the Jacobi identity: For $x, y, z \in \mathbb{R}^3$:

$$\begin{aligned}
&(x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\
&= \quad (x^2y^3 - x^3y^2, x^3y^1 - x^1y^3, x^1y^2 - x^2y^1) \times z \\
&\quad + (y^2z^3 - y^3z^2, y^3z^1 - y^1z^3, y^1z^2 - y^2z^1) \times x \\
&\quad + (z^2x^3 - z^3x^2, z^3x^1 - z^1x^3, z^1x^2 - z^2x^1) \times y \\
&= \quad (x^3y^1z^3 - x^1y^3z^3 - x^1y^2z^2 + x^2y^1z^2, x^1y^2z^1 - x^2y^1z^1 - x^2y^3z^3 + x^3y^2z^3, \\
&\qquad x^2y^3z^2 - x^3y^2z^2 - x^3y^1z^1 + x^1y^3z^1) \\
&\quad + (y^3z^1x^3 - y^1z^3x^3 - y^1z^2x^2 + y^2z^1x^2, y^1z^2x^1 - y^2z^1x^1 - y^2z^3x^3 + y^3z^2x^3, \\
&\qquad y^2z^3x^2 - y^3z^2x^2 - y^3z^1x^1 + y^1z^3x^1) \\
&\quad + (z^3x^1y^3 - z^1x^3y^3 - z^1x^2y^2 + z^2x^1y^2, z^1x^2y^1 - z^2x^1y^1 - z^2x^3y^3 + z^3x^2y^3, \\
&\qquad z^2x^3y^2 - z^3x^2y^2 - z^3x^1y^1 + z^1x^3y^1) \\
&= (0, 0, 0)
\end{aligned}$$

Recall we may write the cross product as the formal determinant:

$$(x^1, x^2, x^3) \times (y^1, y^2, y^3) = \det \begin{bmatrix} i & j & k \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{bmatrix}.$$

So by the properties of det, $\times$ is bilinear and antisymmetric.

Then $\times$ is indeed a Lie bracket and $\mathbb{R}^3$ with $\times$ is a Lie algebra.

8-22. For A an algebra over $\mathbb{R}$, the set of derivations $D : A \to A$, linear maps satisfying $D(xy) = D(x)y + xD(y)$, is equipped with the bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

This set is a real vector space because $(\alpha D_1 + D_2)(xy) = \alpha D_1(xy) + D_2(xy) =$
$= \alpha D_1(x)y + \alpha x D_1(y) + D_2(x)y + xD_2(y) = (\alpha D_1 + D_2)(x)y + x(\alpha D_1 + D_2)(y)$.

It's closed under the bracket because:

$$\begin{aligned} [D_1, D_2](xy) &= D_1(D_2(xy)) - D_2(D_1(xy)) \\ &= D_1(D_2(x)y + xD_2(y)) - D_2(D_1(x)y + xD_1(y)) \\ &= D_1(D_2(x))y + D_2(x)D_1(y) + D_1(x)D_2(y) + xD_1(D_2(y)) \\ &\quad - D_2(D_1(x))y - D_1(x)D_2(y) - D_2(x)D_1(y) - xD_2(D_1(y)) \\ &= [D_1, D_2](x)y + x[D_1, D_2](y) \end{aligned}$$

So $[D_1, D_2]$ is also a derivation.

The bracket is bilinear because derivations are linear, and antisymmetric since $D_1 \circ D_2 - D_2 \circ D_1 = -(D_2 \circ D_1 - D_1 \circ D_2)$.

Finally, the Jacobi identity holds because:

$$\begin{aligned} & [[D_1, D_2], D_3] + [[D_2, D_3], D_1] + [[D_3, D_1], D_2] \\ = \;& D_1 \circ D_2 \circ D_3 - D_2 \circ D_1 \circ D_3 - D_3 \circ D_1 \circ D_2 + D_3 \circ D_2 \circ D_1 \\ & + D_2 \circ D_3 \circ D_1 - D_3 \circ D_2 \circ D_1 - D_1 \circ D_2 \circ D_3 + D_1 \circ D_3 \circ D_2 \\ & + D_3 \circ D_1 \circ D_2 - D_1 \circ D_3 \circ D_2 - D_2 \circ D_3 \circ D_1 + D_2 \circ D_1 \circ D_3 \\ = \;& 0 \end{aligned}$$

So the bracket makes the space of derivations into a Lie algebra.

8-28. Considering the Lie group homomorphism $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$, the induced Lie algebra homomorphism is $\det_* : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{R}$ given by $\det_* B = (\det_* B^L)|_1$
where B^L is the left invariant vector field with $B^L|_I = B$.

Since by Problem 7-4, $(\det_* B^L)|_1 = d(\det)_I B^L|_I = d(\det)_I B = \mathrm{tr}(B)$,
we find that $\det_* = \mathrm{tr}$. (Note this is a Lie algebra homomorphism since
$\mathrm{tr}[A, B] = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0 = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(B)\mathrm{tr}(A) = [\mathrm{tr}(A), \mathrm{tr}(B)]$).

8-31. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subspace.

(a) Suppose $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.
Then if the quotient space $\mathfrak{g}/\mathfrak{h} = \{x+\mathfrak{h} : x \in \mathfrak{g}\}$ has a Lie algebra structure for which the quotient map $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} : x \mapsto x+\mathfrak{h}$ is a Lie algebra homomorphism, then the Lie bracket must be given by $[x+\mathfrak{h}, y+\mathfrak{h}] = [\pi(x), \pi(y)] = \pi([x,y]) = [x,y]+\mathfrak{h}$.

Indeed, such a Lie bracket is well defined, because if $w+\mathfrak{h} = x+\mathfrak{h}$ and $y+\mathfrak{h} = z+\mathfrak{h}$ then $x-w, z-y \in \mathfrak{h}$ and since $\mathfrak{h}$ is an ideal, $[x-w,y], [z-y,x] \in \mathfrak{h}$. Then we find
$[w+\mathfrak{h}, y+\mathfrak{h}] = [w,y]+\mathfrak{h} = [w,y] + ([x-w,y] - [z-y,x]+\mathfrak{h}) =$
$= [x,y] + [x,z-y] + \mathfrak{h} = [x,z]+\mathfrak{h} = [x+\mathfrak{h}, z+\mathfrak{h}]$
which proves well-definedness.

And this operation is actually a proper Lie bracket, because it is bilinear as the original bracket is multilinear. It is antisymmetric as
$[x+\mathfrak{h}, y+\mathfrak{h}] = [x,y]+\mathfrak{h} = -[y,x]+\mathfrak{h} = -([y,x]+\mathfrak{h}) = -[y+\mathfrak{h}, x+\mathfrak{h}]$.
And it satisfies the Jacobi identity as all the $\mathfrak{h}$'s pull out to a single one all the way on the right and the result is the 0 coset.
So $\mathfrak{g}/\mathfrak{h}$ has a unique Lie algebra structure for which π is a homomorphism.

(b) ($\Rightarrow$) If $\mathfrak{h}$ is an ideal, then we have just proven that the quotient map $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is a Lie algebra homomorphism whose kernel is $\mathfrak{h}$. This completes this direction.

($\Leftarrow$) Suppose $\varphi : \mathfrak{g} \to \mathfrak{p}$ is a homomorphism of Lie algebras, and that $\mathfrak{h} = \ker\varphi$. Then φ is a linear map so $\mathfrak{h}$ is a subspace of $\mathfrak{g}$. Suppose $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$.
Then $\varphi([x,y]) = [\varphi(x), \varphi(y)] = [0, \varphi(y)] = 0$, so $[x,y] \in \mathfrak{h} = \ker\varphi$.
Hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. This completes the second direction.