Name: Mason Haberle	**SID: 3032732029**	**Date: 3/29/20**

Math 214, Homework 9.

21-1. Let G be a Lie group acting continuously on a topological manifold M.

Suppose the action map $(\cdot) : G \times M \to M$ is proper.

Then, for any compact $K \subseteq M \times M$, let $\pi_1, \pi_2 : M \times M \to M$ be the projection maps onto the first and second coordinates. By definition both maps are continuous, so $K_1 = \pi_1(K)$ and $K_2 = \pi_2(K)$ are compact subsets of M.

Then $K_1 \times K_2$ is compact and contains K.

Now, let $\Theta : G \times M \to M \times M$ be the map $(g, p) \mapsto (p, g \cdot p)$.

$$\begin{aligned}\text{Then } \Theta^{-1}(K_1 \times K_2) &= \{(g, p) : p \in K_1, g \cdot p \in K_2\} \\ &= \{(g, p) : p \in K_1\} \cap \{(g, p) : g \cdot p \in K_2\} \\ &= (G \times K_1) \cap (\cdot)^{-1}(K_2)\end{aligned}$$

which, as the intersection of a closed set and a compact set, is compact.

Since $M \times M$ is Hausdorff, K is closed, and thus $\Theta^{-1}(K)$ is a closed subset of the compact set $\Theta^{-1}(K_1 \times K_2)$. So $\Theta^{-1}(K)$ is compact.

Thus Θ is proper, and so is the action.

But the action may be proper while the action map is not.

Indeed, let $G = M = \mathbb{R}$, G acting on M by addition.

Then $(\cdot)^{-1}(0) = \{(x, -x) : x \in \mathbb{R}\}$, which is not compact.

So the action map is not proper. However, for $n \in \mathbb{N}$,

$\Theta^{-1}([-n, n] \times [-n, n]) = \{(t, x) : x \in [-n, n], t + x \in [-n, n]\} \subseteq [-2n, 2n] \times [-n, n]$

is bounded, so the preimage of any compact subset of $\mathbb{R}^2$ under Θ is closed and bounded, hence compact. So Θ and the action are proper.

21-5. Suppose a Lie group G acts smoothly and freely on a smooth manifold M.

Suppose that the orbit space M/G has a smooth manifold structure for which the quotient map $\pi : M \to M/G$ is a smooth submersion.

Then for any neighborhood $U \subseteq M/G$ and any smooth local section $\sigma : U \to M$ of π, the map $f : (g, x) \mapsto g \cdot \sigma(x)$ on $G \times U$ is smooth. If $g \cdot \sigma(x) = h \cdot \sigma(y)$ then

$x = \pi(\sigma(x)) = \pi(g \cdot \sigma(x)) = \pi(h \cdot \sigma(y)) = \pi(\sigma(y)) = y$, so $g \cdot \sigma(x) = h \cdot \sigma(x)$

and since the action is free $g = h$ and f is injective.

Also, for $p \in \pi^{-1}(U)$, $\pi(p) \in U$, and $\sigma(\pi(p)) \in \{g \cdot p : g \in G\}$.

Then there is an $h \in G$ such that $f(h, \pi(p)) = h \cdot \sigma(\pi(\varphi)) = p$, and f maps onto $\pi^{-1}(U)$.

As a smooth, bijective map to $\pi^{-1}(U)$, f is a homeomorphism when restricted to any compact subset of $G \times U$. In particular, if p_i is a sequence in $\pi^{-1}(U)$ such that $p_i \to p \in \pi^{-1}(U)$, we may restrict to a neighborhood of $f^{-1}(p)$ with compact closure in $G \times U$ and this compact neighborhood will be mapped homeomorphically to compact neighborhood of p in $\pi^{-1}(U)$. Taking a tail of the sequence which maps into this neighborhood, it pulls back to a convergent sequence in $G \times U$. So $f^{-1}(p_i)$ converges.

Let $p_i \in M$, $g_i \in G$ be sequences such that $p_i \to p$ and $g_i \cdot p_i \to q$. Then picking a local section $\sigma : U \to M$ with $\pi(q) \in U$, we have that $\pi(g_i \cdot p_i) = \pi(p_i) \to \pi(q)$, so that $\pi(g_i \cdot p_i), \pi(p_i)$ are in U for large enough i.

Then $g_i \cdot p_i, p_i \in \pi^{-1}(U)$ for such i, and applying f^{-1} we find sequences $h_i, k_i \in G$ such that $h_i \cdot \sigma(\pi(p_i)) = p_i$, $k_i \cdot \sigma(\pi(p_i)) = g_i \cdot p_i$, and that $(h_i, \pi(p_i))$, $(k_i, \pi(p_i))$ are convergent sequences in $G \times M/G$. Then h_i^{-1} and k_i are convergent sequences in G.

Moreover, $(k_i^{-1} g_i h_i) \cdot \sigma(\pi(\varphi_i)) = (k_i^{-1} g_i) \cdot p_i = k_i^{-1} \cdot (k_i \cdot \sigma(\pi(\varphi_i))) = \sigma(\pi(\varphi_i))$, so by freeness of the action $k_i^{-1} g_i h_i = e$ and $g_i = k_i h_i^{-1}$. The product of convergent sequences is convergent. Thus g_i converges and we have shown that the action is proper.

21-9. Let $\mathbb{Z}$ act on $\mathbb{R}^2$ by $n \cdot (x, y) = (x + n, (-1)^n y)$.

The action is smooth because $\mathbb{Z}$ is zero-dimensional, so we may fix n and vary (x, y) to check smoothness (upon which the action map is affine, hence smooth).

To check freeness, suppose $(x, y) \in \mathbb{R}^2$ and that $n \cdot (x, y) = (x, y)$. Then $x + n = x$, so $n = 0$. Thus only the identity element fixes any (x, y) and the action is free.

To check properness, suppose (x_k, y_k) is a sequence in $\mathbb{R}^2$ which converges, n_k a sequence in $\mathbb{Z}$, and that $n_k \cdot (x_k, y_k)$ converges in $\mathbb{R}^2$. Then $(x_k + n_k, (-1)^{n_k} y_k)$ converges, so in particular $x_k + n_k$ converges. The difference of two convergent sequences in $\mathbb{R}$ is convergent, so $x_k + n_k - x_k = n_k$ is convergent.
By the equivalent characterizations of properness, this shows that the action is proper.

Now, with $\mathbb{Z}$ acting on $\mathbb{R}$ by translation, we can consider $\mathbb{R}^2$ to be the trivial vector bundle over $\mathbb{R}$ by the map $\pi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$. $\mathbb{Z}$ acts smoothly, freely, and properly on $\mathbb{R}$ and $\mathbb{R}^2$, and π is $\mathbb{Z}$-equivariant (since the action in the first variable is just the action on $\mathbb{R}$).

Moreover, when x and n are fixed, the map $y \mapsto (-1)^n y$ is linear. Then we can define the map $\pi' : \mathbb{R}^2/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by $\mathbb{Z} \cdot (x, y) \mapsto \mathbb{Z} \cdot x$, this is well-defined because if $\mathbb{Z} \cdot (x, y) = \mathbb{Z} \cdot (u, v)$ then $x = u + n$ for some n and $\mathbb{Z} \cdot x = \mathbb{Z} \cdot u$.

π is a surjective smooth map, for $x \in \mathbb{R}/\mathbb{Z}$ we have $\pi^{-1}(x) \cong \mathbb{R}$, and we may take local trivializations to be the identity map into $\mathbb{R}^2$ on slices $((-\varepsilon + x, \varepsilon + x) \times \mathbb{R})/\mathbb{Z}$.

This, as a sketch of problem 21-8, shows that the Mobius bundle over $S^1 = \mathbb{R}/\mathbb{Z}$, defined exactly to be $\mathbb{R}^2/\mathbb{Z}$, is a nontrivial smooth rank-1 vector bundle (nontrivial because when one moves from x to $x + 1$, by the action the y coordinate is negated).

Also, by theorem 21.13 the quotient map is a smooth normal covering map. (Since we haven't covered anything around covering maps or covering groups, my assumption is that this part of the question is not required).

21-16. For V an n-dimensional vector space, $K = (k_1, \dots, k_m)$ with $0 < k_1 < \cdots < k_m < n$, we have an action of $\mathrm{GL}(V)$ on $\mathrm{F}_K(V)$ by $A \cdot (S_1, \dots, S_m) = (AS_1, \dots, AS_m)$.

For any $(S_1, \dots, S_m), (T_1, \dots, T_m) \in \mathrm{F}_K(V)$, we may pick bases $\{e_1, \dots, e_{k_1}\}$, $\{f_1, \dots, f_{k_1}\}$ for S_1 and T_1, complete each to bases $\{e_1, \dots, e_{k_2}\}$ and $\{f_1, \dots, f_{k_2}\}$ for S_2 and T_2, and so on until we have bases $\{e_1, \dots, e_n\}$, $\{f_1, \dots, f_n\}$ for V, where for each i from 1 to m, $\{e_1, \dots, e_{k_i}\}$ is a basis for S_i and $\{f_1, \dots, f_{k_i}\}$ is a basis for T_i. Then simply let $A \in \mathrm{GL}(V)$ be the linear map with $A(e_i) = f_i$, so that $AS_i = T_i$ and $A \cdot (S_1, \dots, S_m) = (AS_1, \dots, AS_m) = (T_1, \dots, T_m)$. This shows the action is transitive.

Given a flag $S = (S_1, \dots, S_m)$, the isotropy subgroup $G_S = \{A \in \mathrm{GL}(V) : AS_i = S_i \ \forall i\}$ in the basis $\{e_i\}$ above is given by the matrices $\begin{pmatrix} A_1 & & * & * \\ & A_2 & & * \\ 0 & & \ddots & \\ 0 & 0 & & A_{m+1} \end{pmatrix}$ where A_i is an invertible $(k_i - k_{i-1}) \times (k_i - k_{i-1})$ matrix (with $k_0 = 0, k_{m+1} = n$).

Indeed, such matrices are invertible because their determinant is the product of the determinants of the A_i, which is nonzero, and such matrices fix the flag because $\{e_1, \dots, e_{k_i}\}$ are sent to linear combinations of $\{e_1, \dots, e_{k_i}\}$.

Such a subgroup is closed as the preimage of the zero vector in the map which picks out all matrix elements under the spaces taken up by the A_i.

Then by Theorem 21.20, $\mathrm{F}_K(V)$ has a unique smooth manifold structure which makes the action of $\mathrm{GL}(V)$ smooth, and $\dim \mathrm{F}_K(V) = \dim \mathrm{GL}(V) - \dim G_S =$
$= n^2 - (n^2 - k_1(k_2 - k_1) - k_2(k_3 - k_2) - \cdots - k_m(n - k_m)) = \sum_{i=1}^m k_i(k_{i+1} - k_i)$.

In fact, $\mathrm{F}_K(V)$ is compact (no matter what K is) since $\mathrm{O}(V)$ also acts transitively on $\mathrm{F}_K(V)$, we do the same as when we showed $\mathrm{GL}(V)$ acts transitively but we just normalize the $\{f_i\}$ basis with respect to the $\{e_i\}$ basis (taking an inner product on V for which $\langle e_i, e_j \rangle = \delta_{ij}$). Then by the Homogeneous Space Characterization Theorem, $\mathrm{F}_K(V)$ is a quotient manifold of $\mathrm{O}(V)$, hence compact since $\mathrm{O}(V)$ is compact.

19-1. Suppose D is a smooth involutive distribution on M. Let $\mathcal{J}(D)$ be the space of differential forms on M which annihilate D. First of all, this is an ideal. Indeed, if ω is a p-form which annihilates D and η is any 1-form, then for smooth sections $X_1, \cdots, X_{p+1}$, we have $(\eta \wedge \omega)(X_1, \ldots, X_{p+1}) = \sum_i (-1)^{i+1} \eta(X_i) \omega(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{p+1}) = 0$

So for any form η, we may write η as a sum of wedge products of 1-forms and by associativity of the wedge product $\eta \wedge \omega$ annihilates D. So $\mathcal{J}(D)$ is an ideal.

Moreover, for any p-form $\eta \in \mathcal{J}(D)$ and any neighborhood U in M on which we have local defining forms for D, $\omega^1, \ldots, \omega^{n-k}$,

we have that $\eta = \sum_i \omega^i \wedge \beta^i$ for some $(p-1)$-forms β^i on U by Lemma 19.6.

Then $d\eta = \sum_i d\omega^i \wedge \beta^i + (-1)^{p-1} \omega^i \wedge d\beta^i$.

By the 1-form criterion for involutivity, the $d\omega^i$ annihilate D, so $d\omega^i, \omega^i \in \mathcal{J}(D|_U)$ and $d\eta \in \mathcal{J}(D|_U)$ since $\mathcal{J}(D|_U)$ is an ideal. The holds in a neighborhood of any $p \in M$.

So $d\eta$ annihilates all of D, and $d\eta \in \mathcal{J}(D)$. Thus $\mathcal{J}(D)$ is a differential ideal.

Now, suppose D is just a smooth distribution, and that $\mathcal{J}(D)$ is a differential ideal. Then for any open set $U \subseteq M$, we have that $\mathcal{J}(D|_U)$ is a differential ideal since restricting forms to neighborhoods does not affect their differential (which is locally defined).

Then in particular, for any smooth 1-form ω that annihilates D on U, $\omega \in \mathcal{J}(D|_U)$, so $d\omega \in \mathcal{J}(D|_U)$ and $d\omega$ also annihilates D on U.

This is the 1-form criterion for involutivity. So D is an involutive distribution.