10
Jacobi Fields

Our goal for the remainder of this book is to generalize to higher dimensions some of the geometric and topological consequences of the Gauss–Bonnet theorem. We need to develop a new approach: instead of using Stokes's theorem and differential forms to relate the curvature to global topology as in the proof of the Gauss–Bonnet theorem, we study how curvature affects the behavior of nearby geodesics. Roughly speaking, positive curvature causes nearby geodesics to converge (Figure 10.1), while negative curvature causes them to spread out (Figure 10.2). In order to draw topological consequences from this fact, we need a quantitative way to measure the effect of curvature on a one-parameter family of geodesics.

We begin by deriving the Jacobi equation, which is an ordinary differential equation satisfied by the variation field of any one-parameter family of geodesics. A vector field satisfying this equation along a geodesic is called a Jacobi field. We then introduce the notion of conjugate points, which are pairs of points along a geodesic where some Jacobi field vanishes. Intuitively, if p and q are conjugate along a geodesic, one expects to find a one-parameter family of geodesics that start at p and end (almost) at q.

After defining conjugate points, we prove a simple but essential fact: the points conjugate to p are exactly the points where $\exp_p$ fails to be a local diffeomorphism. We then derive an expression for the second derivative of the length functional with respect to proper variations of a geodesic, called the "second variation formula." Using this formula, we prove another essential fact about conjugate points: No geodesic is minimizing past its first conjugate point.

In the final chapter, we will derive topological consequences of these facts.

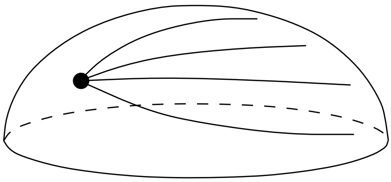

FIGURE 10.1. Positive curvature causes geodesics to converge.

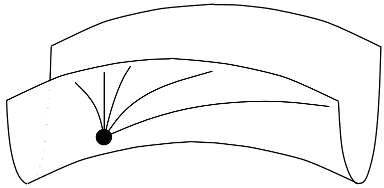

FIGURE 10.2. Negative curvature causes geodesics to spread out.

The Jacobi Equation

In order to study the effect of curvature on nearby geodesics, we focus on variations through geodesics. Suppose therefore that $\gamma\colon [a,b] \to M$ is a geodesic segment, and $\Gamma\colon (-\varepsilon,\varepsilon) \times [a,b] \to M$ is a variation of γ (as defined in Chapter 6). We say Γ is a *variation through geodesics* if each of the main curves $\Gamma_s(t) = \Gamma(s,t)$ is also a geodesic segment. (In particular, this requires that Γ be smooth.) Our first goal is to derive an equation that must be satisfied by the variation field of a variation through geodesics.

Write $T(s,t) = \partial_t\Gamma(s,t)$ and $S(s,t) = \partial_s\Gamma(s,t)$ as in Chapter 6. The geodesic equation tells us that

$$D_t T \equiv 0$$

for all (s,t). We can take the covariant derivative of this equation with respect to s, yielding

$$D_s D_t T \equiv 0.$$

To relate this to the variation field of γ, we need to commute the covariant differentiation operators D_s and D_t. Because these are covariant derivatives acting on a vector field along a curve, we should expect the curvature to be involved. Indeed, we have the following lemma.

Lemma 10.1. *If Γ is any smooth admissible family of curves, and V is a smooth vector field along Γ, then*

$$D_s D_t V - D_t D_s V = R(S,T)V.$$

Proof. This is a local issue, so we can compute in any local coordinates. Writing $V(s,t) = V^i(s,t)\partial_i$, we compute

$$D_t V = \frac{\partial V^i}{\partial t}\partial_i + V^i D_t \partial_i.$$

Therefore,

$$D_s D_t V = \frac{\partial^2 V^i}{\partial s \partial t}\partial_i + \frac{\partial V^i}{\partial t} D_s\partial_i + \frac{\partial V^i}{\partial s} D_t\partial_i + V^i D_s D_t \partial_i.$$

Interchanging D_s and D_t and subtracting, we see that all the terms except the last cancel:

$$D_s D_t V - D_t D_s V = V^i \left(D_s D_t \partial_i - D_t D_s \partial_i \right). \tag{10.1}$$

Now we need to compute the commutator in parentheses. If we write the coordinate functions of Γ as $x^j(s,t)$, then

$$S = \frac{\partial x^k}{\partial s} \partial_k; \qquad T = \frac{\partial x^j}{\partial t} \partial_j.$$

Because ∂_i is extendible,

$$D_t \partial_i = \nabla_T \partial_i = \frac{\partial x^j}{\partial t} \nabla_{\partial_j} \partial_i,$$

and therefore, because $\nabla_{\partial_j} \partial_i$ is also extendible,

$$\begin{aligned} D_s D_t \partial_i &= D_s \left(\frac{\partial x^j}{\partial t} \nabla_{\partial_j} \partial_i \right) \\ &= \frac{\partial^2 x^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial x^j}{\partial t} \nabla_S \left(\nabla_{\partial_j} \partial_i \right) \\ &= \frac{\partial^2 x^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i. \end{aligned}$$

Interchanging $s \leftrightarrow t$ and $j \leftrightarrow k$ and subtracting, we find that the first terms cancel out, and we get

$$\begin{aligned} D_s D_t \partial_i - D_t D_s \partial_i &= \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} \left(\nabla_{\partial_k} \nabla_{\partial_j} \partial_i - \nabla_{\partial_j} \nabla_{\partial_k} \partial_i \right) \\ &= \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} R(\partial_k, \partial_j) \partial_i \\ &= R(S,T) \partial_i. \end{aligned}$$

Finally, inserting this into (10.1) yields the result. □

Theorem 10.2. (The Jacobi Equation) *Let γ be a geodesic and V a vector field along γ. If V is the variation field of a variation through geodesics, then V satisfies*

$$D_t^2 V + R(V, \dot\gamma)\dot\gamma = 0. \tag{10.2}$$

Proof. With S and T as before, the preceding lemma implies

$$\begin{aligned} 0 &= D_s D_t T \\ &= D_t D_s T + R(S,T)T \\ &= D_t D_t S + R(S,T)T, \end{aligned}$$

where the last step follows from the symmetry lemma. Evaluating at $s = 0$, where $S(0,t) = V(t)$ and $T(0,t) = \dot\gamma(t)$, we get (10.2). □

Any vector field along a geodesic satisfying the Jacobi equation is called a *Jacobi field.* Because of the following lemma, which is a converse to Theorem 10.2, each Jacobi field tells us how some family of geodesics behaves, at least "infinitesimally" along γ.

Lemma 10.3. *Every Jacobi field along a geodesic γ is the variation field of some variation of γ through geodesics.*

Exercise 10.1. Prove Lemma 10.3. [Hint: Let $\Gamma(s,t) = \exp_{\sigma(s)} tW(s)$ for a suitable curve σ and vector field W along σ.]

Now we reverse our approach: let's forget about variations for a while, and just study Jacobi fields in their own right. As the following lemma shows, the Jacobi equation can be written as a system of second-order linear ordinary differential equations, so it has a unique solution given initial values for V and D_tV at one point.

Proposition 10.4. (Existence and Uniqueness of Jacobi Fields) *Let $\gamma\colon I \to M$ be a geodesic, $a \in I$, and $p = \gamma(a)$. For any pair of vectors $X, Y \in T_pM$, there is a unique Jacobi field J along γ satisfying the initial conditions*

$$J(a) = X; \qquad D_tJ(a) = Y.$$

Proof. Choose an orthonormal basis $\{E_i\}$ for T_pM, and extend it to a parallel orthonormal frame along all of γ. Writing $J(t) = J^i(t)E_i$, we can express the Jacobi equation as

$$\ddot{J}^i + R_{jkl}{}^i J^j \dot{\gamma}^k \dot{\gamma}^l = 0.$$

This is a linear system of second-order ODEs for the n functions J^i. Making the usual substitution $V^i = \dot{J}^i$ converts it to an equivalent first-order linear system for the $2n$ unknowns $\{J^i, V^i\}$. Then Theorem 4.12 guarantees the existence and uniqueness of a solution on the whole interval I with any initial conditions $J^i(a) = X^i$, $V^i(a) = Y^i$. □

Corollary 10.5. *Along any geodesic γ, the set of Jacobi fields is a $2n$-dimensional linear subspace of $\mathcal{T}(\gamma)$.*

Proof. Let $p = \gamma(a)$ be any point on γ, and consider the map from the set of Jacobi fields along γ to $T_pM \oplus T_pM$ by sending J to $(J(a), D_tJ(a))$. The preceding proposition says precisely that this map is bijective. □

There are always two trivial Jacobi fields along any geodesic, which we can write down immediately (see Figure 10.3). Because $D_t\dot{\gamma} = 0$ and $R(\dot{\gamma},\dot{\gamma})\dot{\gamma} = 0$ by antisymmetry of R, the vector field $J_0(t) = \dot{\gamma}(t)$ satisfies the Jacobi equation with initial conditions

$$J_0(0) = \dot{\gamma}(0); \qquad D_tJ_0(0) = 0.$$

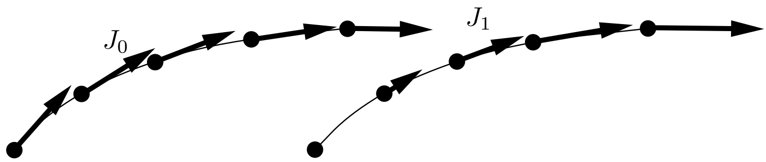

FIGURE 10.3. Trivial Jacobi fields.

Similarly, $J_1(t) = t\dot\gamma(t)$ is a Jacobi field with initial conditions

$$J_1(0) = 0; \qquad D_t J_1(0) = \dot\gamma(0).$$

It is easy to see that J_0 is the variation field of the variation $\Gamma(s,t) = \gamma(s+t)$, while J_1 is the variation field of $\Gamma(s,t) = \gamma(e^s t)$. Therefore, these two Jacobi fields just reflect the possible reparametrizations of γ, and don't tell us anything about the behavior of geodesics other than γ itself.

To distinguish these trivial cases from more informative ones, we make the following definitions. A *tangential vector field* along a curve γ is a vector field V such that $V(t)$ is a multiple of $\dot\gamma(t)$ for all t, and a *normal vector field* is one such that $V(t) \perp \dot\gamma(t)$ for all t.

Lemma 10.6. *Let $\gamma\colon I \to M$ be a geodesic, and $a \in I$.*

(*a*) *A Jacobi field J along γ is normal if and only if*

$$J(a) \perp \dot\gamma(a) \text{ and } D_t J(a) \perp \dot\gamma(a). \tag{10.3}$$

(*b*) *Any Jacobi field orthogonal to $\dot\gamma$ at two points is normal.*

Proof. Using compatibility with the metric and the fact that $D_t\dot\gamma \equiv 0$, we compute

$$\begin{aligned}\frac{d^2}{dt^2}\langle J,\dot\gamma\rangle &= \left\langle D_t^2 J,\dot\gamma\right\rangle\\ &= -\langle R(J,\dot\gamma)\dot\gamma,\dot\gamma\rangle\\ &= -Rm(J,\dot\gamma,\dot\gamma,\dot\gamma) = 0\end{aligned}$$

by the symmetries of the curvature tensor. Thus, by elementary calculus, $f(t) := \langle J(t),\dot\gamma(t)\rangle$ is a linear function of t. Note that $f(a) = \langle J(a),\dot\gamma(a)\rangle$ and $\dot f(a) = \langle D_t J(a),\dot\gamma(a)\rangle$. Thus $J(a)$ and $D_t J(a)$ are orthogonal to $\dot\gamma(a)$ if and only if f and its first derivative vanish at a, which happens if and only if $f \equiv 0$. Similarly, if J is orthogonal to $\dot\gamma$ at two points, then f vanishes at two points and is therefore identically zero. □

As a consequence of this lemma, it is easy to check that the space of normal Jacobi fields is a $(2n-2)$-dimensional subspace of $\mathcal{T}(\gamma)$, and the space of tangential ones is a 2-dimensional subspace. Every Jacobi field can be uniquely decomposed into the sum of a tangential Jacobi field plus a normal Jacobi field, just by decomposing its initial value and initial derivative.

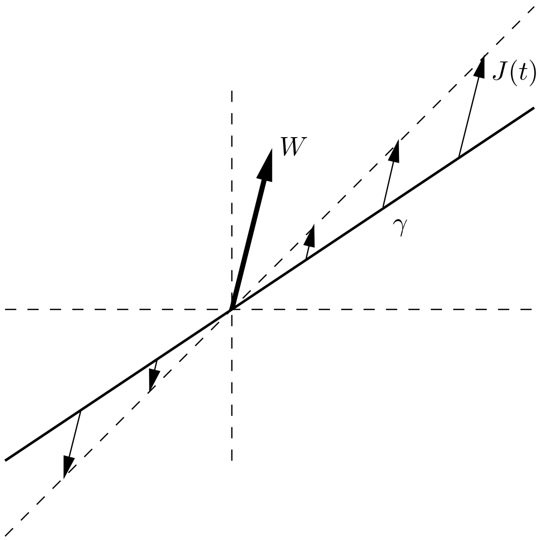

FIGURE 10.4. A Jacobi field in normal coordinates.

Computations of Jacobi Fields

In Riemannian normal coordinates, half of the Jacobi fields are easy to write down explicitly.

Lemma 10.7. *Let $p \in M$, let (x^i) be normal coordinates on a neighborhood $\mathcal{U}$ of p, and let γ be a radial geodesic starting at p. For any $W = W^i \partial_i \in T_pM$, the Jacobi field J along γ such that $J(0) = 0$ and $D_t J(0) = W$ (see Figure 10.4) is given in normal coordinates by the formula*

$$J(t) = tW^i \partial_i. \tag{10.4}$$

Proof. An easy computation using formula (4.10) for covariant derivatives in coordinates shows that J satisfies the specified initial conditions, so it suffices to show that J is a Jacobi field. If we set $V = \dot\gamma(0) \in T_pM$, then we know from Lemma 5.11 that γ is given in coordinates by the formula $\gamma(t) = (tV^1, \ldots, tV^n)$. Now consider the variation Γ given in coordinates by

$$\Gamma(s,t) = (t(V^1 + sW^1), \ldots, t(V^n + sW^n)).$$

Again using Lemma 5.11, we see that Γ is a variation through geodesics. Therefore its variation field $\partial_s \Gamma(0,t)$ is a Jacobi field. Differentiating $\Gamma(s,t)$ with respect to s shows that its variation field is $J(t)$. □

For metrics with constant sectional curvature, we have a different kind of explicit formula for Jacobi fields—this one expresses a Jacobi field as a scalar multiple of a parallel vector field.

Lemma 10.8. *Suppose (M, g) is a Riemannian manifold with constant sectional curvature C, and γ is a unit speed geodesic in M. The normal Jacobi fields along γ vanishing at $t = 0$ are precisely the vector fields*

$$J(t) = u(t)E(t), \tag{10.5}$$

where E is any parallel normal vector field along γ, and $u(t)$ is given by

$$u(t) = \begin{cases} t, & C = 0; \\ R \sin \dfrac{t}{R}, & C = \dfrac{1}{R^2} > 0; \\ R \sinh \dfrac{t}{R}, & C = -\dfrac{1}{R^2} < 0. \end{cases} \tag{10.6}$$

Proof. Since g has constant curvature, its curvature endomorphism is given by the formula of Lemma 8.10:

$$R(X,Y)Z = C\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big).$$

Substituting this into the Jacobi equation, we find that a normal Jacobi field J satisfies

$$\begin{aligned} 0 &= D_t^2 J + C\big(\langle\dot\gamma,\dot\gamma\rangle J - \langle J,\dot\gamma\rangle\dot\gamma\big) \\ &= D_t^2 J + CJ, \end{aligned} \tag{10.7}$$

where we have used the facts that $|\dot\gamma|^2 = 1$ and $\langle J,\dot\gamma\rangle = 0$.

Since (10.7) says that the second covariant derivative of J is a multiple of J itself, it is reasonable to try to construct a solution by choosing a parallel normal vector field E along γ and setting $J(t) = u(t)E(t)$ for some function u to be determined. Plugging this into (10.7), we find that J is a Jacobi field provided u is a solution to the differential equation

$$\ddot u(t) + Cu(t) = 0.$$

It is an easy matter to solve this ODE explicitly. In particular, the solutions satisfying $u(0) = 0$ are constant multiples of the functions given in (10.6). This construction yields all the normal Jacobi fields vanishing at 0, since there is an $(n-1)$-dimensional space of them, and the space of parallel normal vector fields has the same dimension. □

Combining the formulas in the last two lemmas, we obtain our first application of Jacobi fields: explicit expressions for constant curvature metrics in normal coordinates.

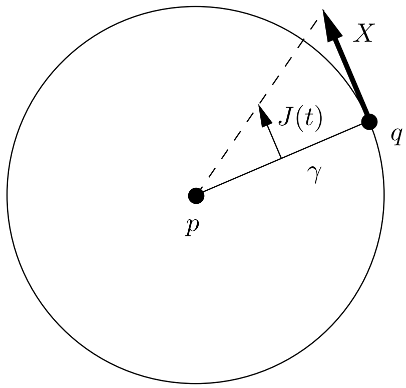

FIGURE 10.5. A vector X tangent to a geodesic sphere is the value of a normal Jacobi field.

Proposition 10.9. *Suppose (M, g) is a Riemannian manifold with constant sectional curvature C. Let (x^i) be Riemannian normal coordinates on a normal neighborhood $\mathcal{U}$ of $p \in M$, let $|\cdot|_{\bar{g}}$ be the Euclidean norm in these coordinates, and let r be the radial distance function. For any $q \in \mathcal{U} - \{p\}$ and $V \in T_qM$, write $V = V^\top + V^\perp$, where $V^\top$ is tangent to the sphere $\{r = \text{constant}\}$ through q and $V^\perp$ is a multiple of $\partial/\partial r$. The metric g can be written*

$$g(V,V) = \begin{cases} |V^\perp|_{\bar{g}}^2 + |V^\top|_{\bar{g}}^2, & K = 0; \\ |V^\perp|_{\bar{g}}^2 + \dfrac{R^2}{r^2}\left(\sin^2 \dfrac{r}{R}\right)|V^\top|_{\bar{g}}^2, & C = \dfrac{1}{R^2} > 0; \\ |V^\perp|_{\bar{g}}^2 + \dfrac{R^2}{r^2}\left(\sinh^2 \dfrac{r}{R}\right)|V^\top|_{\bar{g}}^2, & C = -\dfrac{1}{R^2} < 0. \end{cases} \tag{10.8}$$

Proof. By the Gauss lemma, the decomposition $V = V^\top + V^\perp$ is orthogonal, so $|V|_g^2 = |V^\perp|_g^2 + |V^\top|_g^2$. Since $\partial/\partial r$ is a unit vector in both the g and $\bar{g}$ norms, it is immediate that $|V^\perp|_g = |V^\perp|_{\bar{g}}$. Thus we need only compute $|V^\top|_g$.

Set $X = V^\top$, and let γ denote the unit speed radial geodesic from p to q. By Lemma 10.7, X is the value of a Jacobi field J along γ that vanishes at p (Figure 10.5), namely $X = J(r)$, where $r = d(p,q)$ and

$$J(t) = \frac{t}{r} X^i \partial_i. \tag{10.9}$$

Because J is orthogonal to $\dot{\gamma}$ at p and q, it is normal by Lemma 10.6.

Now J can also be written in the form $J(t) = u(t)E(t)$ as in Lemma 10.8. In this representation,

$$D_t J(0) = \dot{u}(0)E(0) = E(0),$$

since $\dot{u}(0) = 1$ in each of the cases of (10.6). Therefore, since E is parallel and thus of constant length,

$$|X|^2 = |J(r)|^2 = |u(r)|^2|E(r)|^2 = |u(r)|^2|E(0)|^2 = |u(r)|^2|D_tJ(0)|^2. \tag{10.10}$$

Observe that $D_tJ(0) = (1/r)X^i\partial_i|_p$ by (10.9). Since g agrees with $\bar{g}$ at p, we have

$$|D_tJ(0)| = \frac{1}{r}\left|X^i\partial_i\big|_p\right|_g = \frac{1}{r}|X|_{\bar{g}}.$$

Inserting this into (10.10) and using formula (10.6) for $u(r)$ completes the proof. □

Proposition 10.10. (Local Uniqueness of Constant Curvature Metrics) *Let (M, g) and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds with constant sectional curvature C. For any points $p \in M$, $\tilde{p} \in \widetilde{M}$, there exist neighborhoods $\mathcal{U}$ of p and $\widetilde{\mathcal{U}}$ of $\tilde{p}$ and an isometry $F\colon \mathcal{U} \to \widetilde{\mathcal{U}}$.*

Proof. Choose $p \in M$ and $\tilde{p} \in \widetilde{M}$, and let $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ be geodesic balls of small radius ε around p and $\tilde{p}$, respectively. Riemannian normal coordinates give maps $\varphi\colon \mathcal{U} \to B_\varepsilon(0) \subset \mathbf{R}^n$ and $\widetilde{\varphi}\colon \widetilde{\mathcal{U}} \to B_\varepsilon(0) \subset \mathbf{R}^n$, under which both metrics are given by (10.8) (Figure 10.6). Therefore $\widetilde{\varphi}^{-1} \circ \varphi$ is the required local isometry. □

Conjugate Points

Our next application of Jacobi fields is to study the question of when the exponential map is a local diffeomorphism. If (M, g) is complete, we know that $\exp_p$ is defined on all of T_pM, and is a local diffeomorphism near 0. However, it may well happen that it ceases to be even a local diffeomorphism at points far away.

An enlightening example is provided by the sphere $\mathbf{S}^n_R$. All geodesics starting at a given point p meet at the antipodal point, which is at a distance of πR along each geodesic. The exponential map is a diffeomorphism on the ball $B_{\pi R}(0)$, but it fails to be a local diffeomorphism at all points on the sphere of radius πR in $T_p\mathbf{S}^n_R$ (Figure 10.7). Moreover, Lemma 10.8 shows that each Jacobi field on $\mathbf{S}^n_R$ vanishing at p has its first zero precisely at distance πR.

On the other hand, formula (10.4) shows that if $\mathcal{U}$ is a normal neighborhood of p (the image of a set on which $\exp_p$ is a diffeomorphism), no Jacobi field that vanishes at p can vanish at any other point in $\mathcal{U}$. We might thus be led to expect a relationship between zeros of Jacobi fields

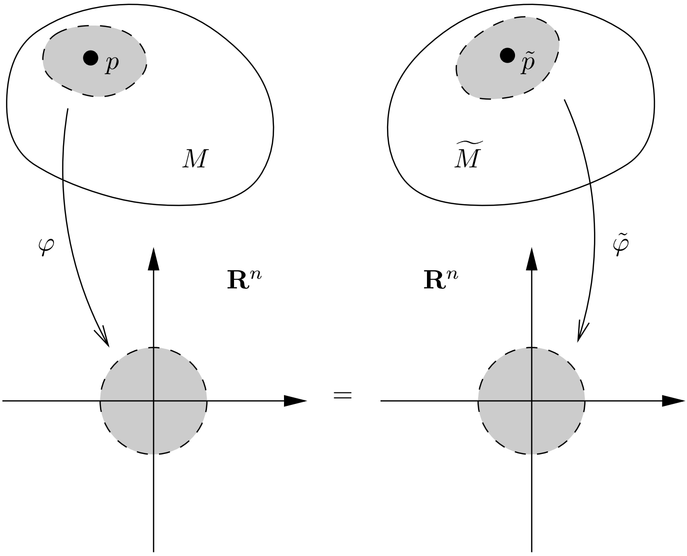

FIGURE 10.6. Local isometry constructed from normal coordinate charts.

and singularities of the exponential map (i.e., points where it fails to be a local diffeomorphism).

If γ is a geodesic segment joining $p, q \in M$, q is said to be *conjugate to p along γ* if there is a Jacobi field along γ vanishing at p and q but not identically zero (Figure 10.8). The *order* or *multiplicity* of conjugacy is the dimension of the space of Jacobi fields vanishing at p and q. From the existence and uniqueness theorem for Jacobi fields, there is an n-dimensional space of Jacobi fields that vanish at p; since tangential Jacobi fields vanish at most at one point, the order of conjugacy of two points p and q can be at most $n-1$. This bound is sharp: Lemma 10.8 shows that if p and q are antipodal points on $\mathbf{S}^n_R$, there is a Jacobi field vanishing at p and q for each parallel normal vector field along γ; thus in that case p and q are conjugate to order exactly $n-1$.

The most important fact about conjugate points is that they are precisely the images of singularities of the exponential map, as the following proposition shows.

Proposition 10.11. *Suppose $p \in M$, $V \in T_pM$, and $q = \exp_p V$. Then $\exp_p$ is a local diffeomorphism in a neighborhood of V if and only if q is not conjugate to p along the geodesic $\gamma(t) = \exp_p tV$, $t \in [0,1]$.*

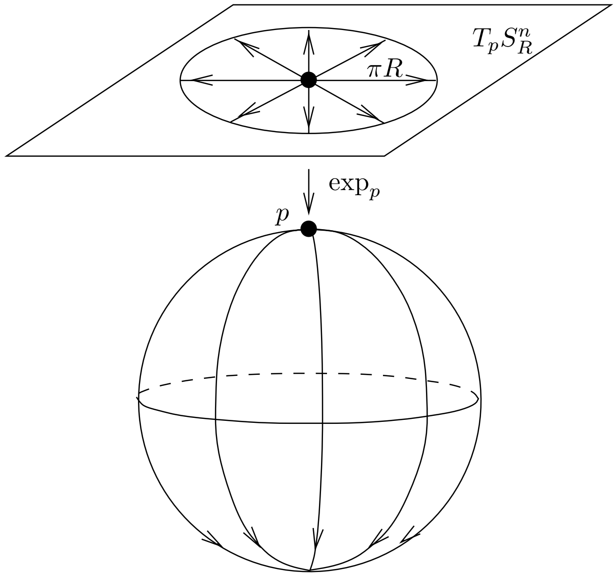

FIGURE 10.7. The exponential map of the sphere.

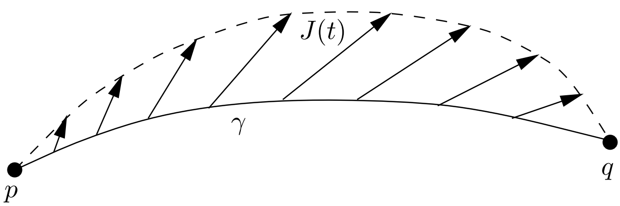

FIGURE 10.8. Conjugate points.

Proof. By the inverse function theorem, $\exp_p$ is a local diffeomorphism near V if and only if $(\exp_p)_*$ is an isomorphism at V, and by dimensional considerations, this occurs if and only if $(\exp_p)_*$ is injective at V.

Identifying $T_V(T_pM)$ with T_pM as usual, we can compute the push-forward $(\exp_p)_*$ at V as follows:

$$(\exp_p)_*W = \left.\frac{d}{ds}\right|_{s=0} \exp_p(V + sW).$$

To compute this, we define a variation of γ through geodesics (Figure 10.9) by

$$\Gamma_W(s,t) = \exp_p t(V + sW).$$

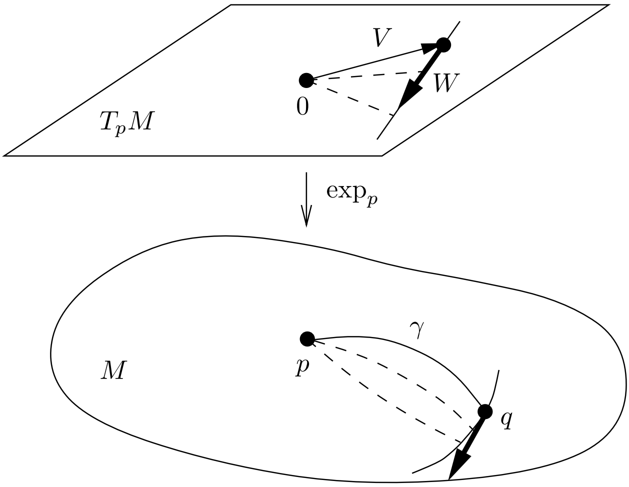

FIGURE 10.9. Computing $(\exp_p)_*W$.

Then the variation field $J_W(t) = \partial_s\Gamma_W(0,t)$ is a Jacobi field along γ, and

$$J_W(1) = (\exp_p)_*W.$$

Since $W \in T_pM$ is arbitrary, there is an n-dimensional space of such Jacobi fields, and so these are all the Jacobi fields along γ that vanish at p. (If γ is contained in a normal neighborhood, these are just the Jacobi fields of the form (10.4) in normal coordinates.)

Therefore, $(\exp_p)_*$ fails to be an isomorphism at V when there is a vector W such that $(exp_p)_*W = 0$, which occurs precisely when there is a Jacobi field J_W along γ with $J_W(0) = J_W(q) = 0$. □

As Proposition 10.4 shows, the "natural" way to specify a unique Jacobi field is by giving its initial value and initial derivative. However, in a number of the arguments above, we have had to construct Jacobi fields along a geodesic γ satisfying $J(0) = 0$ and $J(b) = W$ for some specific vector W. More generally, one can pose the *two-point boundary problem* for Jacobi fields: Given $V \in T_{\gamma(a)}M$ and $W \in T_{\gamma(b)}M$, find a Jacobi field J along γ such that $J(a) = V$ and $J(b) = W$. Another interesting property of conjugate points is that they are the obstruction to solving the two-point boundary problem, as the next exercise shows.

Exercise 10.2. Suppose $\gamma\colon [a,b] \to M$ is a geodesic. Show that the two-point boundary problem for Jacobi fields is uniquely solvable for every pair of vectors $V \in T_{\gamma(a)}M$ and $W \in T_{\gamma(b)}M$ if and only if $\gamma(a)$ and $\gamma(b)$ are not conjugate along γ.

The Second Variation Formula

Our last task in this chapter is to study the question of which geodesics are minimizing. In our proof that any minimizing curve is a geodesic, we imitated the first-derivative test of elementary calculus: If a geodesic γ is minimizing, then the first derivative of the length functional must vanish for any proper variation of γ. Now we imitate the second-derivative test: If γ is minimizing, the second derivative must be nonnegative. First, we must compute this second derivative. In keeping with classical terminology, we call it the *second variation* of the length functional.

Theorem 10.12. (The Second Variation Formula) *Let $\gamma\colon [a,b] \to M$ be a unit speed geodesic, Γ a proper variation of γ, and V its variation field. The second variation of $L(\Gamma_s)$ is given by the following formula:*

$$\left.\frac{d^2}{ds^2}\right|_{s=0} L(\Gamma_s) = \int_a^b \left(\left|D_t V^\perp\right|^2 - Rm(V^\perp, \dot\gamma, \dot\gamma, V^\perp)\right) dt, \tag{10.11}$$

where $V^\perp$ is the normal component of V.

Proof. As usual, write $T = \partial_t \Gamma$ and $S = \partial_s \Gamma$. We begin, as we did when computing the first variation formula, by restricting to a rectangle $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ where Γ is smooth. From (6.3) we have, for any s,

$$\frac{d}{ds} L(\Gamma_s|_{[a_{i-1},a_i]}) = \int_{a_{i-1}}^{a_i} \frac{\langle D_t S, T\rangle}{\langle T, T\rangle^{1/2}} dt.$$

Differentiating again with respect to s, and using the symmetry lemma and Lemma 10.1,

$$\begin{aligned}
&\frac{d^2}{ds^2} L(\Gamma_s|_{[a_{i-1},a_i]}) \\
&= \int_{a_{i-1}}^{a_i} \left(\frac{\langle D_s D_t S, T\rangle}{\langle T, T\rangle^{1/2}} + \frac{\langle D_t S, D_s T\rangle}{\langle T, T\rangle^{1/2}} - \frac{1}{2} \frac{\langle D_t S, T\rangle 2 \langle D_s T, T\rangle}{\langle T, T\rangle^{3/2}} \right) dt \\
&= \int_{a_{i-1}}^{a_i} \left(\frac{\langle D_t D_s S + R(S,T)S, T\rangle}{|T|} + \frac{\langle D_t S, D_t S\rangle}{|T|} - \frac{\langle D_t S, T\rangle^2}{|T|^3} \right) dt.
\end{aligned}$$

Now restrict to $s = 0$, where $|T| = 1$:

$$\begin{aligned}
\left.\frac{d^2}{ds^2}\right|_{s=0} L(\Gamma_s|_{[a_{i-1},a_i]}) = \int_{a_{i-1}}^{a_i} &\big(\langle D_t D_s S, T\rangle - Rm(S,T,T,S) \\
&+ |D_t S|^2 - \langle D_t S, T\rangle^2\big)\, dt\Big|_{s=0}.
\end{aligned} \tag{10.12}$$

Because $D_tT = D_t\dot\gamma = 0$ when $s = 0$, the first term in (10.12) can be integrated as follows:

$$\begin{aligned}\int_{a_{i-1}}^{a_i} \langle D_tD_sS, T\rangle dt &= \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial t}\langle D_sS, T\rangle dt \\ &= \langle D_sS, T\rangle \bigg|_{t=a_{i-1}}^{t=a_i}.\end{aligned} \tag{10.13}$$

Notice that $S(s,t) = 0$ for all s at the endpoints $t = a_0 = a$ and $t = a_k = b$ because Γ is a proper variation, so $D_sS = 0$ there. Moreover, along the boundaries $\{t = a_i\}$ of the smooth regions, $D_sS = D_s(\partial_s\Gamma)$ depends only on the values of Γ when $t = a_i$, and it is smooth up to the line $\{t = a_i\}$ from both sides; therefore D_sS is continuous for all (s,t). Thus when we insert (10.13) into (10.12) and sum over i, the boundary contributions from the first term all cancel, and we get

$$\begin{aligned}\frac{d^2}{ds^2}\bigg|_{s=0} L(\Gamma_s) &= \int_a^b \left(|D_tS|^2 - \langle D_tS, T\rangle^2 - Rm(S,T,T,S)\right) dt\bigg|_{s=0} \\ &= \int_a^b \left(|D_tV|^2 - \langle D_tV, \dot\gamma\rangle^2 - Rm(V,\dot\gamma,\dot\gamma,V)\right) dt.\end{aligned} \tag{10.14}$$

Any vector field V along γ can be written uniquely as $V = V^\top + V^\perp$, where $V^\top$ is tangential and $V^\perp$ is normal. Explicitly,

$$V^\top = \langle V, \dot\gamma\rangle\dot\gamma; \qquad V^\perp = V - V^\top.$$

Because $D_t\dot\gamma = 0$, it follows that

$$D_tV^\top = \langle D_tV, \dot\gamma\rangle\dot\gamma = (D_tV)^\top; \qquad D_tV^\perp = (D_tV)^\perp.$$

Therefore,

$$|D_tV|^2 = |(D_tV)^\top|^2 + |(D_tV)^\perp|^2 = \langle D_tV, \dot\gamma\rangle^2 + |D_tV^\perp|^2.$$

Also,

$$Rm(V,\dot\gamma,\dot\gamma,V) = Rm(V^\perp,\dot\gamma,\dot\gamma,V^\perp)$$

because $Rm(\dot\gamma,\dot\gamma,\cdot,\cdot) = Rm(\cdot,\cdot,\dot\gamma,\dot\gamma) = 0$. Substituting these relations into (10.14) gives (10.11). □

It should come as no surprise that the second variation depends only on the normal component of V; intuitively, the tangential component of V contributes only to a reparametrization of γ, and length is independent of parametrization. For this reason, we generally apply the second variation formula only to variations whose variation fields are proper and normal.

We define a symmetric bilinear form I, called the *index form,* on the space of proper normal vector fields along γ by

$$I(V,W) = \int_a^b (\langle D_t V, D_t W\rangle - Rm(V,\dot\gamma,\dot\gamma,W))\, dt. \tag{10.15}$$

You should think of $I(V,W)$ as a sort of "Hessian" or second derivative of the length functional. Because every proper normal vector field along γ is the variation field of some proper variation, the preceding theorem can be rephrased in terms of the index form in the following way.

Corollary 10.13. *If Γ is a proper variation of a unit speed geodesic γ whose variation field is a proper normal vector field V, the second variation of $L(\Gamma_s)$ is $I(V,V)$. In particular, if γ is minimizing, then $I(V,V) \geq 0$ for any proper normal vector field along γ.*

The next proposition gives another expression for I, which makes the role of the Jacobi equation more evident.

Proposition 10.14. *For any pair of proper normal vector fields V, W along a geodesic segment γ,*

$$I(V,W) = -\int_a^b \left\langle D_t^2 V + R(V,\dot\gamma)\dot\gamma, W\right\rangle dt - \sum_{i=1}^k \langle \Delta_i D_t V, W(a_i)\rangle\,, \tag{10.16}$$

where $\{a_i\}$ are the points where V is not smooth, and $\Delta_i D_t V$ is the jump in $D_t V$ at $t = a_i$.

Proof. On any subinterval $[a_{i-1}, a_i]$ where V and W are smooth,

$$\frac{d}{dt}\langle D_t V, W\rangle = \left\langle D_t^2 V, W\right\rangle + \langle D_t V, D_t W\rangle\,.$$

Thus, by the fundamental theorem of calculus,

$$\int_{a_{i-1}}^{a_i} \langle D_t V, D_t W\rangle\, dt = -\int_{a_{i-1}}^{a_i} \left\langle D_t^2 V, W\right\rangle + \left.\langle D_t V, W\rangle\right|_{a_{i-1}}^{a_i}.$$

Summing over i, and noting that W is continuous at $t = a_i$ and $W(a) = W(b) = 0$, we get (10.16). □

Geodesics Do Not Minimize Past Conjugate Points

In this section, we use the second variation to prove another extremely important fact about conjugate points: No geodesic is minimizing past its

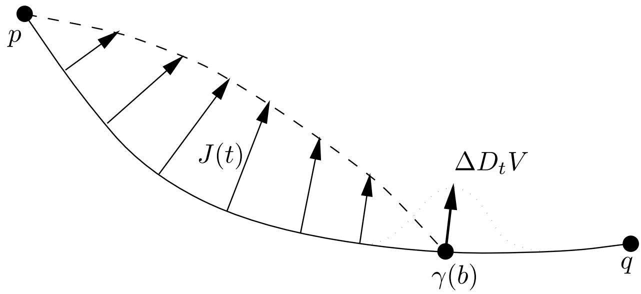

FIGURE 10.10. Constructing a vector field X with $I(X, X) < 0$.

first conjugate point. The geometric intuition is as follows: Suppose γ is minimizing. If $q = \gamma(b)$ is conjugate to $p = \gamma(a)$ along γ, and J is a Jacobi field vanishing at p and q, there is a variation of γ through geodesics, all of which start at p. Since $J(q) = 0$, we can expect them to end "almost" at q. If they really did all end at q, we could construct a broken geodesic by following some Γ_s from p to q and then following γ from q to $\gamma(b + \varepsilon)$, which would have the same length and thus would also be a minimizing curve. But this is impossible: as the proof of Theorem 6.6 shows, a broken geodesic can always be shortened by rounding the corner.

The problem with this heuristic argument is that there is no guarantee that we can construct a variation through geodesics that actually end at q. The proof of the following theorem is based on an "infinitesimal" version of rounding the corner to obtain a shorter curve.

Theorem 10.15. *If γ is a geodesic segment from p to q that has an interior conjugate point to p, then there exists a proper normal vector field X along γ such that $I(X, X) < 0$. In particular, γ is not minimizing.*

Proof. Suppose $\gamma\colon [0, b] \to M$ is a unit speed parametrization of γ, and $\gamma(a)$ is conjugate to $\gamma(0)$ for some $0 < a < b$. This means there is a nontrivial normal Jacobi field J along $\gamma|_{[0,a]}$ that vanishes at $t = 0$ and $t = a$. Define a vector field V along all of γ by

$$V(t) = \begin{cases} J(t), & t \in [0, a]; \\ 0, & t \in [a, b]. \end{cases}$$

This is a proper, normal, piecewise smooth vector field along γ.

Let W be a smooth proper normal vector field along γ such that $W(b)$ is equal to the jump $\Delta D_t V$ at $t = b$ (Figure 10.10). Such a vector field is easily constructed in local coordinates and extended to all of γ by a bump function. Note that $\Delta D_t V = -D_t J(b)$ is not zero, because otherwise J would be a Jacobi field satisfying $J(b) = D_t J(b) = 0$, and thus would be identically zero.

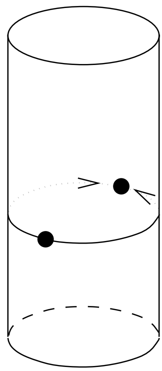

FIGURE 10.11. Geodesics on the cylinder.

For small positive ε, let $X_\varepsilon = V + \varepsilon W$. Then

$$\begin{aligned} I(X_\varepsilon, X_\varepsilon) &= I(V + \varepsilon W, V + \varepsilon W) \\ &= I(V,V) + 2\varepsilon I(V,W) + \varepsilon^2 I(W,W). \end{aligned}$$

Since V satisfies the Jacobi equation on each subinterval $[0,a]$ and $[a,b]$, and $V(a) = 0$, (10.16) gives

$$I(V,V) = -\langle \Delta D_t V, V(a) \rangle = 0.$$

Similarly,

$$I(V,W) = -\langle \Delta D_t V, W(b) \rangle = -|W(b)|^2.$$

Thus

$$I(X_\varepsilon, X_\varepsilon) = -2\varepsilon |W(b)|^2 + \varepsilon^2 I(W,W).$$

If we choose ε small enough, this is strictly negative. □

There is a far-reaching quantitative generalization of Theorem 10.15 called the Morse index theorem, which we do not treat here. The *index* of a geodesic segment is defined to be the maximum dimension of a linear space of proper normal vector fields on which I is negative definite. Roughly speaking, the index is the number of independent directions in which γ can be deformed to decrease its length. (Analogously, the index of a critical point of a function on $\mathbf{R}^n$ is defined as the number of negative eigenvalues of its Hessian.) The Morse index theorem says that the index of any geodesic segment is finite, and is equal to the number of its interior conjugate points counted with multiplicity. (Proofs can be found in [CE75], [dC92], or [Spi79, volume 4].)

It is important to note, by the way, that the converse of Theorem 10.15 is not true: a geodesic without conjugate points need not be minimizing. For example, on the cylinder $\mathbf{S}^1 \times \mathbf{R}$, there are no conjugate points along any geodesic; but no geodesic that wraps more than halfway around the cylinder is minimizing (Figure 10.11). Therefore it is useful to make the following definitions. Suppose γ is a geodesic starting at p. Let $B = \sup\{b > 0 : \gamma|_{[0,b]}$ is minimizing$\}$. If $B < \infty$, we call $q = \gamma(B)$ the *cut point* of p along γ. The *cut locus* of p is the set of all points $q \in M$ such that q is the cut point of p along some geodesic. (Analogously, the *conjugate locus* of p is the set of points q such that q is the first conjugate point to p along some geodesic.) The preceding theorem can be interpreted as saying that the cut point (if it exists) occurs at or before the first conjugate point along any geodesic.

Problems

10-1. Extend the result of Lemma 10.8 by finding a formula for *all* normal Jacobi fields in the constant curvature case, not just the ones that vanish at 0.

10-2. Suppose that all sectional curvatures of M are nonpositive. Use the results of this chapter to show that the conjugate locus of any point is empty. [We will give a more geometric proof in the next chapter.]

10-3. Suppose (M, g) is a Riemannian manifold and $p \in M$. Show that the second-order Taylor series of g in normal coordinates centered at p is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{kl} R_{iklj}x^k x^l + O(|x|^3).$$

[Hint: Let $\gamma(t) = (tV^1, \ldots, tV^n)$ be a radial geodesic and $J(t) = tW^i\partial_i$ a Jacobi field along γ, and compute the first four t-derivatives of $|J(t)|^2$ at $t = 0$ in two ways.]