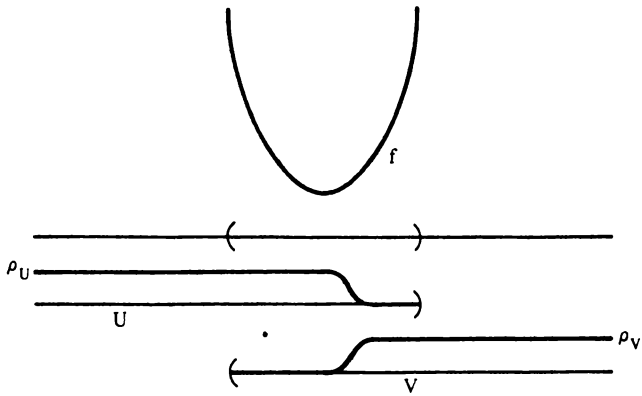

Figure 2.1

we see that $\Omega^0(U) \oplus \Omega^0(V) \to \Omega^0(\mathbb{R}^1)$ is surjective. For a general manifold M, if $\omega \in \Omega^q(U \cap V)$, then $(-\rho_V \omega, \rho_U \omega)$ in $\Omega^q(U) \oplus \Omega^q(V)$ maps onto ω. □

The Mayer-Vietoris sequence

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

induces a long exact sequence in cohomology, also called a Mayer-Vietoris sequence:

$$\text{(2.4)} \qquad \begin{array}{l} \to H^{q+1}(M) \to H^{q+1}(U) \oplus H^{q+1}(V) \to H^{q+1}(U \cap V) \to \\ \qquad \qquad \qquad \qquad \nwarrow d^* \\ \to H^q(M) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \end{array}$$

We recall again the definition of the coboundary operator d^* in this explicit instance. The short exact sequence gives rise to a diagram with exact rows

$$\begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 \to & & \Omega^{q+1}(M) & \to & \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) & \to & \Omega^{q+1}(U \cap V) & & \to 0 \\ & & d\uparrow & & d\uparrow & & d\uparrow & & \\ 0 \to & & \Omega^q(M) & \to & \Omega^q(U) \oplus \Omega^q(V) & \to & \Omega^q(U \cap V) & & \to 0 \\ & & & & \cup\!\!\!\!\!\!\; & & \cup\!\!\!\!\!\!\; & & \\ & & & & \xi & & \omega & & d\omega = 0 \end{array}$$

Let $\omega \in \Omega^q(U \cap V)$ be a closed form. By the exactness of the rows, there is a $\xi \in \Omega^q(U) \oplus \Omega^q(V)$ which maps to ω, namely, $\xi = (-\rho_V \omega, \rho_U \omega)$. By the commutativity of the diagram and the fact that $d\omega = 0$, $d\xi$ goes to 0 in $\Omega^{q+1}(U \cap V)$, i.e., $-d(\rho_V \omega)$ and $d(\rho_U \omega)$ agree on the overlap $U \cap V$. Hence $d\xi$ is the image of an element in $\Omega^{q+1}(M)$. This element is easily seen to be closed and represents $d^*[\omega]$. As remarked earlier, it can be shown that $d^*[\omega]$ is independent of the choices in this construction. Explicitly we see that the coboundary operator is given by

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U \\ [d(\rho_U \omega)] & \text{on } V. \end{cases} \tag{2.5}$$

We define the *support* of a form ω on a manifold M to be $\operatorname{Supp} \omega = \{ p \in M \mid \omega(p) \neq 0 \}$. Note that in the Mayer-Vietoris sequence $d^*\omega \in H^*(M)$ has support in $U \cap V$.

EXAMPLE 2.6 (The cohomology of the circle). Cover the circle with two open sets U and V as shown in Figure 2.2. The Mayer-Vietoris sequence gives

$$\begin{array}{lcccc} & S^1 & & U \amalg V & & U \cap V \\ H^2 & 0 & & 0 & & 0 \\ H^1 & & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & d^* & & \\ H^0 & & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\ \delta\ } & \mathbb{R} \oplus \mathbb{R} \end{array}$$

The difference map δ sends (ω, τ) to $(\tau - \omega, \tau - \omega)$, so $\operatorname{im} \delta$ is 1-dimensional. It follows that $\ker \delta$ is also 1-dimensional. Therefore,

$$H^0(S^1) = \ker \delta = \mathbb{R}$$

$$H^1(S^1) = \operatorname{coker} \delta = \mathbb{R}.$$

We now find an explicit representative for the generator of $H^1(S^1)$. If $\alpha \in \Omega^0(U \cap V)$ is a closed 0-form which is not the image under δ of a closed form in $\Omega^0(U) \oplus \Omega^0(V)$, then $d^*\alpha$ will represent a generator of $H^1(S^1)$. As α we may take the function which is 1 on the upper piece of $U \cap V$ and 0 on

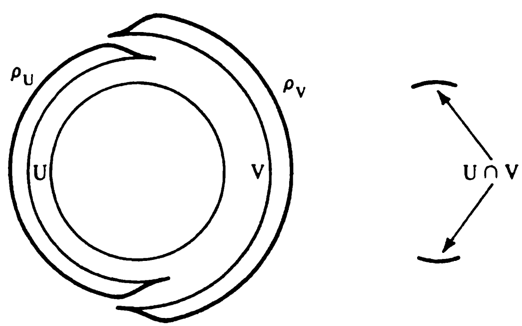

Figure 2.2