A. $36 \quad q: x \rightarrow y$ open quotient map.
$Y$ is Hausdorff $\Leftrightarrow \underline{R} \subset X \times X$ is closed.
Recall: quotient map:

$$
x \rightarrow x / \sim=
$$

$\sim$ : equivalence relation: symmotior. transitive.

$$
\left\{\begin{array}{l}
x \sim y, \Leftrightarrow y \sim x \\
x \sim y, y \sim z \Rightarrow x \sim z \\
x \sim x \\
R \subset x \times x \\
=\{(x, y) \mid x \sim y\} .
\end{array}\right.
$$

$Y=X / \sim$ Hausdorff.
means $\forall y_{1} \neq y_{2} \in Y$, we wont to find open $u_{1} \ni y_{1}, u_{2} \nRightarrow y_{2} . \quad u_{1} \cap u_{2} \neq \phi$. assume $R$ is closed.
$y_{1}, y_{2}$ are two distinct classes

$x_{1} \in y_{1}, \quad x_{2} \in y_{2}$ representatives.


$$
x_{1}+x_{2} \Leftrightarrow\left(x_{1}, x_{2}\right) \in R
$$

$\because R$ is closed.
$\therefore \exists$ open set $V_{1} \times V_{2} \ni\left(x_{1}, x_{2}\right)$

$$
V_{1} \times V_{2} \cap R=\varnothing .
$$



$$
\begin{aligned}
U_{i} & =\left\{x \in X \mid \quad \exists \tilde{x}_{i} \in V_{i}, \quad x \sim x_{i}\right\} \\
& =\left[V_{i}\right] .
\end{aligned}
$$

so $\quad u_{1} \cap U_{2}=\phi$

$$
U_{1}=\pi\left(V_{1}\right)^{\subset Y}, U_{2}=\pi\left(V_{2}\right) \subset Y .
$$

$\because \pi$ is open $\therefore U_{1}, U_{2}$ are open
done $($ in $\Leftarrow) \quad U_{1}, U_{2}$ disjoint open around $y_{1}, y_{2}$.
$(\Rightarrow)$ exercise.
A. 57 If $F: X \rightarrow Y . \quad Y$ locally compact if $F$ is proper and continuous. Hauridoff. then $F$ is closed.

Pf: Given $K \subset X$ closed. wis $F(K)$ is closed. Suppose $\left\{y_{i}\right\}$ is a sep of point in $F(K), \quad y_{i} \rightarrow y, \quad y \in Y$.
wis. $y \in F(K)$.

- using LCH of $Y$, we have a precompact nitid $u$ of $y$, i.e. $\bar{u}$ is compact.
- using $F$ is proper, $F^{-1}(\bar{u})$ is compact.
$\mathcal{F}^{-1}(\bar{u}) \cap K$ is compact.

$\Rightarrow F\left(K \cap F^{-1}(\bar{U})\right)$ is compact. $F(k) \cup U$.

$$
\begin{aligned}
& y \in \underset{\tau_{\text {using compactness. }}}{F\left(M \cap F^{-1}(\bar{u})\right)} \subset F(K) \text {. } \\
& \therefore \quad y \in F(K) \Rightarrow F(K) \text { is closed. }
\end{aligned}
$$

Back to Cl 21.
Lie topompl.
Prop 21.4: $\quad G C M$ continually \& property
then $M / G$ is Hausdorff:
Pf: By def of proper action:

$$
\begin{aligned}
\theta: \quad M \times G & \rightarrow M \times M \\
(p, g) & \mapsto(p, g \cdot p)
\end{aligned}
$$

is proper
$\therefore$ Relation subset $R=\operatorname{Im}(\Theta)$ is.
closed. ( $\because$ is $\Theta$ is $\quad$ coper continuous $\Rightarrow$ closal $\Rightarrow \operatorname{Im}(\theta)$ is closed).

Using A. $36: \sum_{\pi}^{\text {since }} \pi \quad M \rightarrow M / G$ is open.
and $R$ is closed
$\therefore M / G$ is Hausdorff.

(a) action is proper. i.e.

$$
G \times M \rightarrow M \times M \quad \text { is proper }
$$

(b) $\quad P_{i} \rightarrow P \quad$ in $M$.

$$
g_{i} \cdot p_{i} \rightarrow p^{\prime} \text { in } M \text {. }
$$

then
$g_{i}$ sub. converge to some $g \in G$.
(c). Given any $K \subset M$. compact
lat $G_{k}=\{g \in G \mid g K \cap K \neq \phi\}$.
then $G_{k}$ is compact.


- $\quad P_{i} \rightarrow p \quad, \quad g_{i} p_{i} \rightarrow p^{\prime}$

$$
\left(p_{i}, g_{-} P_{i}\right) \rightarrow\left(p, p^{\prime}\right) .
$$

to use (a). We will need to produce a compact subset.
let $u \ni p, u^{\prime} \ni p^{\prime}$ be precompact open $\bar{u}^{\prime} \times \bar{u} \Rightarrow$ (pip') is compact.
$\left(p_{i}, g_{i}\right) \in \underbrace{\Theta^{-1}\left(\bar{u}^{\prime} \times \bar{u}\right)}$. compact.

G
so $g_{i}$ wandering in

$$
\pi_{G} \Theta^{-1}\left(\overline{U^{\prime}} \times \bar{U}\right) .
$$

a compact set
so it subconverges.
(b) $\Rightarrow(c)$
(b):

(c)


WIS $G_{k}$ is compact take aseq $g_{i} \in G_{k}$. WT $g_{i} \cdots$ g

- promote seq of $g_{i}$. to a seq. $f$ ares with endpoints in $R$

$$
\left(\frac{p_{i}}{\hat{k}}, \frac{g_{i} \cdot p_{i}}{k}\right)
$$

- now using compares of $K$., assume

$$
p_{i} \rightarrow P, \quad g_{i} \cdot p_{i} \rightarrow p^{\prime}
$$

now use $(b) \Rightarrow g_{i} \cdots g$.
(c) $\Rightarrow$ (a)

given L $C M \times M$ pt, ned $\Theta^{-1}(2)$ is compact.

$$
\begin{aligned}
& K=\pi_{1}(L) \cup \underline{\pi_{2}(L)} \quad \text { compact. } \\
& \begin{array}{l}
(p, g \cdot p) \in L
\end{array} \\
& \frac{\Theta^{-1}(L)}{\text { closed }} \subset \underbrace{\pi_{1}(L) \times G_{K}} \text { compact } \\
& \Rightarrow \theta^{-1}(L) \text { compact. }
\end{aligned}
$$

Sketch of the proof of Quotient Mfd Thy:


- Key: $\mathcal{O}_{\text {create transversal }}$ slice. $Y$.
(2) $Y \cap$ with each orbit at most once.
(1) $T M$ has a "distribution" $D_{p} \subseteq T_{p} M$

- distribution is involutive a.priovi.
$\Rightarrow$ involutive distribution has so called flat coordinates.

$$
\begin{aligned}
& u, \varphi \\
& \varphi=\left(\underline{x}^{x_{1}, \cdots, x_{n}}, y_{1}, \cdots, y_{k}\right)
\end{aligned}
$$

st. Distribution: is gen by

$$
\left\{\partial x_{1}, \cdots, \partial x_{n}\right\}
$$

$\Rightarrow \begin{gathered}\exists \text { transversal slice. } y^{y} \uparrow \neq \neq \\ (\text { by fix } x \text {. let } y \text { ) }\end{gathered}$ change.

