Name (Last, First): ______________________________

Student ID: ______________________________

Circle your section:

201	Shin	8am	71 Evans	212	Lim	1pm	3105 Etcheverry
202	Cho	8am	75 Evans	213	Tanzer	2pm	35 Evans
203	Shin	9am	105 Latimer	214	Moody	2pm	81 Evans
204	Cho	9am	254 Sutardja Dai	215	Tanzer	3pm	206 Wheeler
205	Zhou	10am	254 Sutardja Dai	216	Moody	3pm	61 Evans
206	Theerakarn	10am	179 Stanley	217	Lim	8am	310 Hearst
207	Theerakarn	11am	179 Stanley	218	Moody	5pm	71 Evans
208	Zhou	11am	254 Sutardja Dai	219	Lee	5pm	3111 Etcheverry
209	Wong	12pm	3 Evans	220	Williams	12pm	289 Cory
210	Tabrizian	12pm	9 Evans	221	Williams	3pm	140 Barrows
211	Wong	1pm	254 Sutardja Dai	222	Williams	2pm	220 Wheeler

If none of the above, please explain: ______________________________

This is a closed book exam, no notes allowed. It consists of 8 problems, each worth 10 points. We will grade all 8 problems, and count your top 6 scores.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total Possible	60	

Problem 1) True or False. Decide if each of the following statements is TRUE or FALSE. You do not need to justify your answers. Write the full word **TRUE** or **FALSE** in the answer box of the chart. (Each correct answer receives 2 points, incorrect answers or blank answers receive 0 points.)

Statement	1	2	3	4	5
Answer					

1) For any inner product on $\mathbb{R}^2$, if vectors $\mathbf{u}, \mathbf{v}$ satisfy $\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 1$ and $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

2) In the vector space of continuous functions on the interval $[-1, 1]$ with inner product

$$\langle f(t), g(t) \rangle = \int_{-1}^{1} f(t)g(t)\, dt$$

the functions $\cos(t)$ and $\sin(t)$ are orthogonal.

3) If A is symmetric and U is orthogonal, then UAU^{-1} is symmetric.

4) If a 2×2 matrix A has eigenvalues λ_1, λ_2, then its characteristic polynomial is equal to

$$\chi_A(t) = t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2$$

$= C^{\infty}(\mathbb{R})$

5) Let V be the vector space of differentiable functions on the real line. The linear transformation $T : V \to V$ given by $T(y) = y'' - e^{-t}y' + 2y$ is injective.

False.

Q : can we find $y(x)$ non zero, such that

$$y'' - e^{-x} \cdot y' + 2y = 0$$

$\Downarrow$

this equation will have non trivial sol'n.

$y_0(x) = y(x)$.

has sol'n.

$y_1(x) = y'(x)$

$y_1' = y'' = e^{-x}y' - 2y = e^{-x}\cdot y_1 - 2y_0$

$$\boxed{\frac{d}{dx}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & e^{-x} \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}$$

Problem 2) Multiple Choice. There is a single correct answer to each of the following questions. Determine what it is and write the letter in the answer box of the chart. You do not need to justify your answers. (Each correct answer receives 2 points, incorrect answers or blank answers receive 0 points.)

Question	1	2	3	4	5
Answer					

1) Which of the following matrices is similar to $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$?

A) $\begin{bmatrix} -4 & 1 \\ 0 & 5 \end{bmatrix}$ *B)* $\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ *C)* $\begin{bmatrix} 5 & 1 \\ 0 & -4 \end{bmatrix}$ *D)* $\begin{bmatrix} 1 & 6 \\ 0 & -2 \end{bmatrix}$ *E)* none of the preceding.

2) For some basis B of the vector space $\mathbb{R}^2$, the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ have coordinates $[\mathbf{u}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $[\mathbf{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is the vector $\mathbf{w}$ with coordinates $[\mathbf{w}]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

A) $\begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$ *B)* $\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$ *C)* $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ *D)* $\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$ *E)* not determined by the data.

3) For which pair of real numbers (a, b) is the matrix $\begin{bmatrix} 1 & -2 & -1 \\ -1 & a & 1 \\ 3 & -6 & b \end{bmatrix}$ rank one?

$A)$ $(-1,-3)$ $B)$ $(2,-1)$ $C)$ $(2,-3)$ $D)$ $(-2,3)$ $E)$ none of the preceding.

4) What is the sum of the dimensions of the null space and column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}?$$

$A)$ 4 $B)$ 5 $C)$ 6 $D)$ 7 $E)$ 8

5) For which triples of real numbers (a, b, c) does the linear system

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a & 0 & -1 \\ 1 & b & c \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

have a solution for any $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$?

$A)$ $(0,1,2)$ $B)$ $(2,1,0)$ $C)$ $(2,2,1)$ $D)$ $(1,0,2)$ $E)$ none of the preceding.

Problem 3) 1) (5 points) Find the orthogonal projection of the vector $\mathbf{b}$ to the subspace of $\mathbb{R}^4$ spanned by $\mathbf{u}, \mathbf{v}$ where

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

2) (5 points) Find a least-squares approximate solution to the equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 4) 1) (5 points) Find the general solution of the second order ODE

$$y'' - 2y' - 3y = 0$$

(quick way): plug in $y(x) = e^{\lambda x}$,

$$(\lambda^2 - 2\lambda - 3) \cdot e^{\lambda x} = 0$$

$$\Leftrightarrow \quad \lambda^2 - 2\lambda - 3 = 0 \qquad \text{(char polynomial)}$$

$$\Leftrightarrow \quad (\lambda - 1)^2 = 4.$$

$$\Leftrightarrow \quad \lambda - 1 = \pm 2$$

$$\lambda = \begin{cases} 3 \\ -1. \end{cases}$$

two distinct eigenvalues

gen sol'n $y(x) = \underline{c_1 e^{-1 \cdot x} + c_2 \cdot e^{3x}}.$

2) (5 points) Find the general solution of the second order ODE

$$y'' - 2y' - 3y = 10\cos(t)$$

strategy: find a particular sol'n first. (Recall the exercise 3.7.3 in ODE).

$$10\cos(t) = 10 \cdot \left(\frac{e^{it} + e^{-it}}{2} \right)$$

$$= 5 \cdot e^{it} + 5 \cdot e^{-it}.$$

Find $y_1(t)$, s.t.

$$P = \left[\left(\frac{d}{dt}\right)^2 - 2\frac{d}{dt} - 3 \right]$$

(*) $\underbrace{P \; y_1(t) = 5 \cdot e^{it}}$

(**) $P \; y_2(t) = 5 \; e^{-it}.$

then $y_1(t) + y_2(t)$ will be a particular sol'n

ansatz: $y_1(t) = c_1 \cdot e^{it}$, plug in

(*) $(i^2 - 2i - 3) \cdot c \cdot e^{it} = 5 \cdot e^{it}$

$$\therefore \; c_1 = \frac{5}{-1 - 2i - 3} = \frac{5}{-4 - 2i}$$

$y_2(t) = c_2 \cdot e^{-it}$, plug in

(**)

$\Rightarrow \quad C_2 = \dfrac{5}{-4+2i}$

$y_p(t) = y_1(t) + y_2(t) = \dfrac{5}{-4-2i} e^{it} + \dfrac{5e^{-it}}{-4+2i}$

Gen $y_p(t) + b_1 e^{-t} + b_2 e^{3t}$ b_1, b_2 free

Problem 5) 1) (5 points) Find a basis of solutions of the equation

$$\mathbf{y}'(t) = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \mathbf{y}(t)$$

- Find Jordan decomposition of A

$\det(A - \lambda) = 0$

$\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 2^2 = 0$

$\Rightarrow \quad 1-\lambda = \pm 2$

$\Rightarrow \quad \lambda = 1 \pm 2 = \begin{cases} 3 \\ -1 \end{cases}$

eigenvector : $\lambda_1 = 3$, $(A-\lambda) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$.

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \ker(A) = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1$

$\lambda_2 = -1$, $\quad A - \lambda = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, $\quad \ker = \text{span}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_2$

$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\begin{matrix} v_1 \cdot e^{\lambda_1 t} \\ v_2 \cdot e^{\lambda_2 t} \end{matrix}}$ basis of sol'n.

let $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then $A \cdot \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{C} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$C^{-1} \cdot A \cdot C = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = J.$, $\quad A = C \cdot J \cdot C^{-1}$

column of this space is gen sol'n.

$(e^{At}) = C \cdot e^{Jt} \cdot C^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot C^{-1}$

$$= \Big[\underbrace{\begin{pmatrix}1\\1\end{pmatrix} \cdot e^{3t}}, \; \underbrace{\begin{pmatrix}1\\-1\end{pmatrix} e^{-t}} \Big] \cdot C^{-1}.$$

2) (5 points) Write down a 3×3 matrix A such that the equation $\mathbf{y}'(t) = A\mathbf{y}(t)$ has a basis of solutions

$$\mathbf{y}_1(t) = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} 0 \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}_3(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$V_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \lambda_1 = -1$$

$$V_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \lambda_2 = 2$$

$$V_3 = \begin{pmatrix}0\\1\\-1\end{pmatrix}, \quad \lambda_3 = 0.$$

$$C = \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

$$\underline{A} \cdot C = C \cdot \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}.$$

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$A = C \cdot \begin{pmatrix} 1 & & \\ & 2 & \\ & & 0 \end{pmatrix} \cdot C^{-1}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \quad \cdots$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Problem 6) (10 points) Use separation of variables to find a solution $u = u(x,t)$ of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u$$

satisfying $u(x,0) = e^x$ and $u(x,t) \to 0$ as $t \to \infty$ for all x.

set $u(t,x) = g(t) \cdot f(x)$. then

$$\left(\frac{\partial}{\partial t}\right)^2 u = (\partial_t^2 g) \cdot f(x)$$

$$(\partial_x^2) u = g \cdot (\partial_x^2 f).$$

$$(\partial_t^2 g) \cdot f = g \cdot (\partial_x^2 f) + g \cdot f.$$

$$\frac{(\partial_t^2 g)}{g} = \frac{\partial_x^2 f}{f} + 1. = \lambda.$$

$$\begin{cases} \partial_t^2 g = \lambda \cdot g \\ \partial_x^2 f = (\lambda - 1) \cdot f. \end{cases} \Rightarrow \begin{pmatrix} g_\lambda = e^{\sqrt{\lambda} t} c_1 + e^{-\sqrt{\lambda} t} c_2 \end{pmatrix}$$

$$\begin{pmatrix} f_\lambda = e^{\sqrt{\lambda - 1}\, x} c_3 + e^{-\sqrt{\lambda-1}\, x} c_4. \end{pmatrix}$$

gen sol'n $\sum_\lambda C_\lambda \cdot g_\lambda(t) \cdot f_\lambda(x).$

$f(x) \cdot g(0) = e^{x}$. , Let $\lambda = 2$,

$f_2(x) = e^{x}$. , $g_2(x) = e^{-\sqrt{2}t}$. , one sol'n $\boxed{e^{-\sqrt{2}t} \cdot e^{x}}$ satisfies both requirements.

Problem 7) Consider the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

for the function $|x|$ on the interval $[-\pi, \pi]$.

1) (5 points) Calculate the coefficients a_n, for all n.

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right].$$

$$\int_{-\pi}^{\pi} (- -) \cdot \cos(n_0 x)\, dx.$$

$$\int_{-\pi}^{\pi} (RHS) \cos(n_0 x)\, dx$$

$$= \int_{-\pi}^{\pi} a_n \cdot \cos(n_0 x) \cos(n_0 x)\, dx.$$

$$= \int_{-\pi}^{\pi} \cdot a_n \cdot \left(\frac{e^{in_0 x} + e^{-in_0 x}}{2} \right)^2 dx$$

$$= \int_{-\pi}^{\pi} a_n \frac{1}{4} \left(1 + 1 + e^{2in_0 x} + e^{-2in_0 x} \right) dx$$

$$= a_n \cdot 2\pi \cdot \frac{1}{2} = \pi \cdot a_n.$$

$$\int^{\pi} \ldots \int^{\pi} x \cos(n_0 x)\, dx$$

$$\cdot \int_{-\pi} |x| \cos(n_0 x)\, dx = 2 \cdot \int_0 x \cdot \cos(n_0 x)\, dx.$$

$$= 2 \cdot \int_0^{\pi} x \cdot d\left(\frac{\sin(n_0 x)}{n_0}\right).$$

2) (5 points) Calculate the coefficients b_n, for all n.

$$= 2 \cdot x \cdot \frac{\sin n_0 x}{n_0} \Bigg|_{x=0}^{x=\pi}$$

$$- 2 \cdot \int_0^{\pi} \frac{\sin(n_0 x)}{n_0} \cdot dx$$

$$= -2 \int_0^{\pi} \frac{\sin(n_0 x)}{n_0}\, dx$$

$$= -\frac{2}{n_0} \left(\frac{\cos n_0 x}{-n_0}\right)\Bigg|_0^{\pi}$$

$$\boxed{\pi a_n = +\frac{2}{n_0^2}\left((-1)^{n_0} - 1\right)}$$

$$\left(\cos(n_0 \pi) = (-1)^{n_0}\right)$$

$$a_n = \frac{2}{\pi \cdot n_0^2}\left((-1)^{n_0} - 1\right).$$

Problem 8) The following assertions are FALSE. Provide a counterexample (4 points each) along with a clear and brief justification no longer than one sentence (1 point each).

1) (5 points) If A is a 2×2 symmetric matrix with positive integer entries, then any eigenvalue of A is positive or zero.

2) (5 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is an injective linear transformation. For any given basis of $\mathbb{R}^3$, there is a basis of $\mathbb{R}^2$ such that the matrix of T takes the form

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$