Jordan Normal Form.

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- Let $\mathbb{K}$ be any field ($\mathbb{Q}$, $\mathbb{C}$, or $\mathbb{F}_q$).
  Let V be a finite dim v.s. over $\mathbb{K}$.

Let $T: V \to V$ be a linear operator on V.

we want to classify T up to "similarity transf"

We say T and $\tilde{T}$ are similar, if $\exists\ V \xrightarrow{C} V$ invertible transformation, such that $T = C \cdot \tilde{T} \cdot C^{-1}$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "C"', "\simeq"] & V \arrow[d, "C", "\simeq"'] \\
V \arrow[r, "\tilde{T}"] & V
\end{tikzcd}$$

In other words, we want to find a nice basis of V, such that T " looks as diagonal as possible ".

$n = \dim_{\mathbb{K}} V$

- characteristic polynomial on $T : V \to V$.

$$\det(\lambda \cdot I - T) = \lambda^n + p_1 \cdot \lambda^{n-1} + \cdots + p_n.$$

determinant for a linear operator $A : V \to V$:

- pick any basis $e_1, \cdots, e_n$ of V,
  then A become a matrix, such that

$$A e_i = A_{i1} e_1 + \cdots + A_{in} e_n.$$

$$\det(A) := \det([A]).$$

$[A]$ matrix with entries

A_{ij}

- if we choose a different basis, say $\tilde{e}_1, \cdots, \tilde{e}_n$. Then the corresponding matrix

$$[\tilde{A}] = C\ [A]\cdot C^{-1}$$

thus
$$\begin{aligned}\det([\tilde{A}]) &= \det(C\cdot[A]\cdot C^{-1})\\ &= \det(C)\cdot\det([A])\cdot\det(C^{-1})\\ &= \det(C)\cdot\det(C)^{-1}\cdot\det([A]\\ &= \det([A]).\end{aligned}$$

- Assume $\lambda_0 \in \mathbb{K}$ is a root of the characteristic polynomial, i.e. $\det(\lambda_0 I - T) = 0$.

  - If $x \in V$ satisfies
    $$T\cdot x = \lambda_0 \cdot x$$
    and $x \neq 0$, then we say x is an eigenvector of T with eigenvalue λ_0.

  - $\ker(\lambda_0 I - T)$: eigenspace of T with eigenvalue λ_0.
    $= \{x \in V \mid Tx = \lambda_0 x\}$.

  - root space (or generalized eigenspace). of eigenvalue λ_0.

Consider $T - \lambda_0 I : V \to V$.

and its power $(T-\lambda_0 I)^2$, $(T-\lambda_0 I)^3$, ---

if $(T - \lambda_0 I)^k \cdot x = 0$, then.

$$(T-\lambda_0 I)^{k+1} x = (T-\lambda_0 I) \cdot \underbrace{(T-\lambda_0 I)^k \cdot x}_{-} = 0$$

$$\ker(T-\lambda_0) \subset \ker((T-\lambda_0)^2) \subset \cdots$$

$$\subset \ker((T-\lambda_0 I)^k) \subset \ker((T-\lambda_0 I)^{k+1}) \subset \cdots$$

Since they are all subspaces of V, and V is finite dimensional, we know for certain m,

$$W_{\lambda_0} := \ker((T-\lambda_0)^m) = \ker((T-\lambda_0)^{m+1}) = \cdots$$

$\uparrow$ root space. for λ_0.

$$\mathcal{U}_{\lambda_0} := \operatorname{im}((T-\lambda_0)^m)$$

Lemma :
- W_{λ_0} and $\mathcal{U}_{\lambda_0}$ are T-invariant.
- $W_{\lambda_0} \cap \mathcal{U}_{\lambda_0} = \{0\}$.
- $V = W_{\lambda_0} \oplus \mathcal{U}_{\lambda_0}$.

Pf : ① If $v \in W_{\lambda_0}$, then $(T-\lambda_0)^m \cdot v = 0$.

$$(T-\lambda_0 I)^m (Tv) = T\cdot (T-\lambda_0 I)^m \cdot v = 0.$$

$\Rightarrow Tv \in W_{\lambda_0}$

If $v \in U_{\lambda_0}$, then $\exists\, \tilde{v} \in V$, s.t.

$v = (T-\lambda_0 I)^m \cdot \tilde{v}$;

Then $Tv = T\cdot (T-\lambda_0 I)^m \cdot \tilde{v}$

$= (T-\lambda_0 I)^m \cdot (T\tilde{v}).$

$\therefore\ Tv \in U_{\lambda_0}.$

②. Suppose $0 \neq v \in U_{\lambda_0} \cap W_{\lambda_0}$, then.

$V = (T-\lambda_0 I)^m \cdot \tilde{v}$

and $(T-\lambda_0 I)^m \cdot v = 0$.

$\Rightarrow\ (T-\lambda_0 I)^m \cdot (T-\lambda_0 I)^m \cdot \tilde{v} = 0$

$\Rightarrow\ \tilde{v} \in \ker((T-\lambda_0 I)^{2m})$ but

$\tilde{v} \notin \ker((T-\lambda_0 I)^m)$

this contradicts with $\ker((T-\lambda_0 I)^m) = \ker((T-\lambda_0 I)^{m+1})$

$= \cdots$

Thus, $\{0\} = U_{\lambda_0} \cap W_{\lambda_0}.$

③. Recall for any lin map

$A: V_1 \to V_2$,

we have $\dim V_1 = \dim \ker A + \dim \operatorname{im} A.$

apply this to

$(T-\lambda_0 I)^m : V \to V$

we get

$$\dim V = \dim W_{\lambda_0} + \dim U_{\lambda_0}.$$

- Recall that for any 2 vector subspace $V_1, V_2 \subset V$.

we have

$$\dim(V_1+V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Thus. $$\dim(W_{\lambda_0}+U_{\lambda_0}) = \dim W_{\lambda_0} + \dim U_{\lambda_0} \underbrace{- \dim(W_{\lambda_0} \cap U_{\lambda_0})}_{= 0 \quad \text{by } ②}.$$

- Thus $W_{\lambda_0} + U_{\lambda_0} \subset V$

  and $\dim(W_{\lambda_0} + U_{\lambda_0}) = \dim V$

  Hence $$V = W_{\lambda_0} + U_{\lambda_0} = W_{\lambda_0} \oplus U_{\lambda_0}. \qquad \#.$$

Assume $\det(\lambda I - T) = (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_r)^{m_r}$

Lemma: $V = W_{\lambda_1} \oplus W_{\lambda_2} \oplus \cdots \oplus W_{\lambda_r}.$

Pf: $V = W_{\lambda_1} \oplus U_{\lambda_1}$ by prev Lemma.

$W_{\lambda_1} = \ker((T-\lambda_1)^{N_1})$, $U_{\lambda_1} = \operatorname{im}((T-\lambda_1)^{N_1})$

claim: $W_{\lambda_2}, \cdots, W_{\lambda_r} \subset U_{\lambda_1}.$

indeed, if we consider

$T - \lambda_1 \cdot I$ restricted to W_{λ_2}.

it preserves W_{λ_2}. And $\forall\, 0 \neq v \in W_{\lambda_2}$,

$(T-\lambda_1)\, v \neq 0$, (if $Tv = \lambda_1 v$, then $(T-\lambda_2)^{N_2} v = \underline{(\lambda_1-\lambda_2)}^{N_2} \cdot v \neq 0$, contradiction).

Thus $T-\lambda_1|_{W_{\lambda_2}}$ is invertible.

$(T-\lambda_1)^{N_1}|_{W_{\lambda_2}}$ is invertible.

$$W_{\lambda_2} = (T-\lambda_1)^{N_1}(W_{\lambda_2}) \subset (T-\lambda_1)^{N_1}(V) = U_{\lambda_1}$$

This claim shows $W_{\lambda_i} \cap W_{\lambda_j} = \{0\}$.

Hence

$$V = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_r} \oplus U.$$

$U = U_{\lambda_1} \cap \cdots \cap U_{\lambda_r}$, hence is preserved by T

claim: If $V = V_1 \oplus V_2$, and $T : V \to V$ preserves V_1 and V_2, then.

$$\mathrm{char}(T) = \mathrm{char}(T|_{V_1}) \cdot \mathrm{char}(T|_{V_2}).$$

pf: one can choose a basis of V, adapted to $V = V_1 \oplus V_2$, then T matrix is block diagonal

$$T = \begin{array}{cc} \begin{matrix} V_1 & V_2 \end{matrix} & \\ \left[\begin{array}{c|c} T_1 & 0 \\ \hline 0 & T_2 \end{array}\right] & \begin{matrix} V_1 \\ V_2. \end{matrix} \end{array}$$

$$\det(\lambda - T) = \det\left(\begin{array}{c|c} \lambda - T_1 & 0 \\ \hline 0 & \lambda - T_2 \end{array}\right) = \det(\lambda - T_1) \cdot \det(\lambda - T_2).$$

$$\underset{\substack{/\!/ \\ (\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_r)^{m_r}.}}{\operatorname{char}(T)} = \overset{\substack{(\lambda-\lambda_1) \\ \|}}{\operatorname{char}(T|_{W_{\lambda_1}})} \cdot \overset{\substack{(\lambda-\lambda_2) \\ \|}}{\operatorname{char}(T|_{W_{\lambda_2}})} \cdots \overset{---.}{\operatorname{char}(T|_{W_{\lambda_r}})} \cdot \underbrace{\operatorname{char}(T|_{\mathcal{U}})}_{\substack{/\!/ \\ 1.}}$$

has no roots in $\lambda_1, \cdots, \lambda_r$.

Thus $\mathcal{U} = \{0\}$, $\quad n_i = m_i \qquad \#$

- $T-\lambda_i I$ on W_{λ_i} is a <u>nilpotent operator</u>.
  i.e. $(T-\lambda_i I)^N = 0$ on W_{λ_i} for N large enough.

- Ex: $\quad N : \mathbb{K}^n \to \mathbb{K}^n$, $\quad e_1, \cdots, e_n$ be std basis,
  $N(e_n) = e_{n-1}$, $\quad N(e_k) = e_{k-1}$,
  $\cdots \quad N(e_1) = 0$.

$$N = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & \vdots & & \ddots & 1 \\ 0 & 0 & & & 0 \end{pmatrix}$$

such N satisfies

$$\boxed{N^n = 0, \quad \text{but} \quad N^{n-1} \neq 0.}$$

we call such operator on $\mathbb{K}^n$ a regular nilpotent operator.

$$N : \quad e_n \mapsto e_{n-1} \mapsto e_{n-2} \mapsto \cdots \mapsto e_1 \mapsto 0$$

Prop: Let V be an n-dim v.s. $/ \mathbb{K}$.
Let $N : V \to V$ be a nilpotent op
then there exist a decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

such that $N|_{V_k}$ is a regular nilpotent op.

i.e. there exist a basis of V, s.t.

$$N = \left[\begin{array}{cccc} n_1 & n_2 & & \\ \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}} & 0 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}} & & \\ 0 & & \ddots & \end{array}\right]$$

of block diagonal form, where each diagonal block is regular nilpotent.

Pf: (method 1). Assume $N^m = 0$, $N^{m-1} \neq 0$.
Then consider $\mathrm{im}(N^k)$, If $v \in \mathrm{Im}(N^k)$,
then $v = N^k \cdot (\tilde{v}) = N^j (N^{k-j} v)$ for any $j < k$
Thus $\mathrm{im}(N^j) \supset \mathrm{im}(N^k)$ for any $j < k$.

$$V \supset \mathrm{im}(N) \supset \mathrm{im}(N^2) \supset \cdots \supset \mathrm{im}(N^{m-1}) \supset \mathrm{im}(N^m) = 0.$$

We are going to construct a basis adapted to this flag, in the following sense.

For $\mathrm{im}(N^{m-1})$, we find a basis.

$\mathrm{im}(N^{m-1})$:

$\mathrm{im}(N^{m-1})$.

$$\begin{tikzcd}
e_1 , & e_2 , & \cdots , & e_k . & & & \\
e_1^{(2)} \arrow[u] & e_2^{(2)} \arrow[u] & & e_k^{(2)} . \arrow[u] & e_{k+1}^{(2)} & \cdots & e_{k+n_2}^{(2)} . \\
e_1^{(3)} \arrow[u] & e_2^{(3)} \arrow[u] & & \vdots & \boxed{e^{(3)}} \arrow[u] & & \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots & \vdots \arrow[u] & & \\
e_1^{(m)} \arrow[u] & e_2^{(m)} \arrow[u] & & \vdots & \boxed{e_{k+1}^{(m)}} \arrow[u] & &
\end{tikzcd}$$

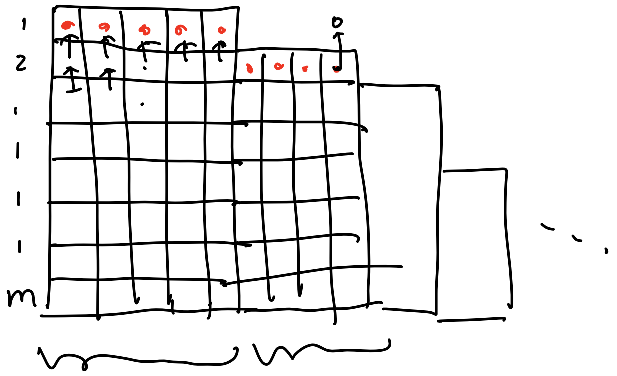
m-1 arrows.

$\mathrm{im}(N^{m-2})$. may contain more than

$\mathrm{span}(\underbrace{e_1, \cdots, e_k, e_1^{(1)}, \cdots, e_k^{(1)}})$

we can complete ↓ to a basis of $\mathrm{im}(N^{m-2})$

by adding $e_{k+1}^{(2)} \cdots e_{k+n_2}^{(2)}$.



due to $\mathrm{im}(N^{m-1}$

This way, we have a basis of V, compatible with the action of N.

Each column of basis vectors generate a ~~such~~ subspace V_i., where N acts on V_i regular nilpotently.

#.

Jordan Normal Form Thm.

Let $\mathbb{K} = \mathbb{C}$., $V = \mathbb{C}^n$.

Let $T : V \to V$.

Then., there exists a basis of V, such that T is in block diagonal form, where each block is

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}.$$

<u>Pf</u>: we first decompose

$$\det(\lambda - T) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_r)^{m_r}$$

and define root spaces W_{λ_i}.

Then $(T - \lambda_i)$ on W_{λ_i} is a nilp operator.

Hence W_{λ_i} decomposes into.

$$W_{\lambda_i}^{(1)} \oplus \cdots \oplus W_{\lambda_i}^{(n_i)}$$

where $T - \lambda_i$ acts on each block reg. nilp.

$$(T-\lambda_i)\Big|_{W_{\lambda_i}} = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\;} \end{pmatrix}$$

$$T\Big|_{W_{\lambda_i}} = \begin{pmatrix} \boxed{\begin{matrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\;} \end{pmatrix}$$

and similarly for all roots spaces.

$$\therefore\ V = W_{\lambda_1}^{(1)} \oplus \cdots \oplus W_{\lambda_1}^{(n_1)} \oplus W_{\lambda_2}^{(1)} \oplus \cdots \oplus W_{\lambda_2}^{(1)} \oplus \cdots$$