1. Existence and Uniqueness of sol'n.

- let $x_1(t), \cdots, x_n(t)$ be the unknown function. $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$
- let
$$\begin{cases} \dot{x}_1(t) = F_1(t, x_1, \cdots, x_n) \\ \dot{x}_2(t) = F_2(t, x_1, \cdots, x_n) \\ \vdots \\ \dot{x}_n(t) = F_n(t, x_1, \cdots, x_n) \end{cases}$$
be the equations.

<u>Thm</u>: Suppose $\{F_i(t, \vec{x})\}$ have continuous (partial) derivatives in t and $x_1, \cdots, x_n$, then for any initial condition at $t = t_0$, given by $(x_1(t_0), \cdots, x_n(t_0))$, we have a small open segment $(t_0 - \varepsilon, t_0 + \varepsilon)$ containing t, such that the sol'n $\vec{x}(t)$ exists for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ that satisfies the eqn & initial condition. Such sol'n is unique.

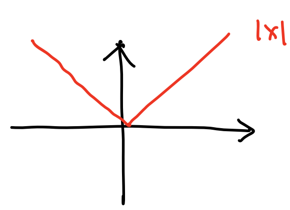
<u>Ex</u>: a function that does not have continous derivative:

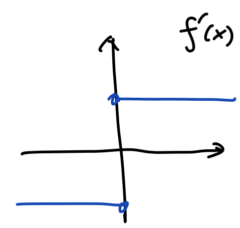
- $f(x) = \frac{1}{x}$ $\quad x \in (-\infty, +\infty)$.

  function is not continuous at $x = 0$

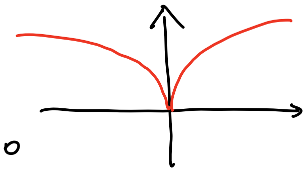
  $\Rightarrow f'(x)$ doesnot exist at $x = 0$

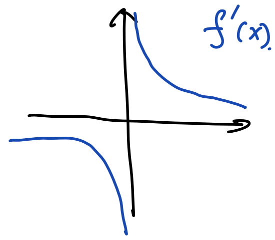
- $f(x) = |x|$





- $f(x) = |x|^{\frac{1}{2}}$





$$f'(x) = \begin{cases} \frac{1}{2}\frac{1}{x} & x > 0 \\ -\frac{1}{2}\frac{1}{x} & x < 0. \end{cases}$$

- Ex :
$$\begin{cases} \dot{x}(t) = [x(t)]^{\frac{2}{3}} & \text{diff eq.} \\ x(0) = 0 & \text{initial condition} \end{cases}$$

we have 2 sol'n.

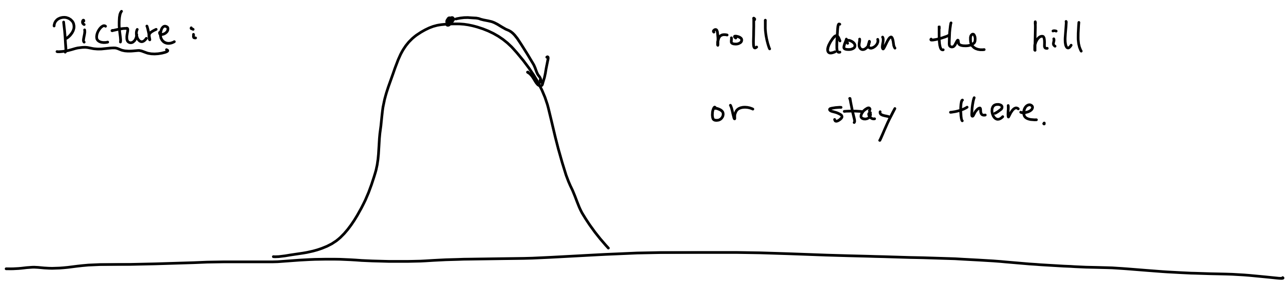
$$x(t) = 0 \qquad \forall\, t.$$

or

$$x(t) = \left(\frac{t}{3}\right)^3 \quad \Rightarrow \quad \dot{x}(t) = \left(\frac{t}{3}\right)^2$$

We don't have uniqueness here, since $F(t, x) = x^{2/3}$ doesn't have continuous derivative at $x = 0$.

Picture :



roll down the hill

or stay there.

(2)

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ \vdots & \cdots & \cdots & \vdots \end{bmatrix}}_{A} \begin{pmatrix} x(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$\{a_{ij}\}$ are constants.

const coeff

coeff is indep of t.

Homogeneous. First Order. Ordinary Diff Eq.

all terms in eq are linear in $\vec{x}$

i.e. if we change $\vec{x} \rightsquigarrow 2\vec{x}$, that term change by factor of 2.

only $\frac{d}{dt}$ appears (no $\left(\frac{d}{dt}\right)^2$)

$x_i(t)$ only depends on one variable t.

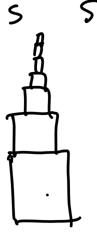
$$\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t) \Rightarrow \boxed{\vec{x}(t) = e^{A \cdot t}\, \vec{x}(0)}$$

?

Here, since A is an $n \times n$ matrix with constant coeff.

$$e^x = 1 + x + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4!} + \cdots\cdots$$

(this series converges for all $x \in \mathbb{C}$.).



if each term has size that shrinks fast enough, then the total size is finite.

$$e^{A \cdot t} = I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots\cdots$$

- each term make sence $\because \frac{(At)^n}{n!} = \frac{t^n}{n!} \cdot \underbrace{A^n}_{\text{well defined}}$.
- summation converges. (need proof)

$$\frac{d}{dt}\left(e^{A \cdot t}\right) = 0 + A + A^2 \cdot t + A^3 \cdot \frac{t^2}{2!} + \cdots$$

$$= A\cdot\left(I + At + A^2\frac{t^2}{2!} + \cdots\right)$$
$$= A\cdot e^{At}$$

$$\boxed{\frac{d}{dt}e^{\lambda t} = \lambda\cdot e^{\lambda t}}$$

$$\frac{d}{dt}\Big(\underbrace{e^{At}}\Big) = A\cdot e^{At} \quad \text{then}$$

size $n\times n$ matrix

depends on t

$$\frac{d}{dt}\Big(\underbrace{e^{At}\cdot\vec{x}_0}_{\vec{x}(t)}\Big) = A\cdot\underbrace{e^{At}\cdot\vec{x}_0}_{\vec{x}(t)}$$

$$\Rightarrow \quad \frac{d}{dt}\vec{x}(t) = A\cdot\vec{x}(t)$$

<u>notation</u>: e^{At} is called the fundamental sol'n. since, each <u>column vector</u> in e^{At} is a sol'n, the general sol'n $\vec{x}(t) = e^{At}\vec{x}_0$ is a linear comb. of the column's

In practice, to do exponentiation of a matrix A, we want to diagonalize A first.

<u>Recall</u> <u>Jordan Decomposition Thm</u>:

Given a square matrix A over $\mathbb{C}$ of size n, there exist an invertible matrix C, s.t.

$$A = C\cdot J\cdot C^{-1}$$

where J is block diagonal, with each block $\sim \begin{pmatrix}\lambda & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{pmatrix}$

$$A^2 = (C\cdot J \underbrace{C^{-1})\cdot(C}\cdot J\cdot C^{-1})$$
$$= C\cdot J\cdot J\cdot C^{-1} = C\cdot J^2\cdot C^{-1}$$

$\vdots$

$$A^n = (C\cdot J C^{-1})(CJC^{-1})\cdots(C\cdot J C^{-1})$$
$$= C\cdot J^n\cdot C^{-1}$$

$$e^{At} = 1 + At + A^2\cdot\frac{t^2}{2!} + \cdots$$
$$= C\left[1 + Jt + J^2\cdot\frac{t^2}{2!} + \cdots\right]\cdot C^{-1}$$
$$= C\cdot e^{Jt}\cdot C^{-1}$$

$$\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right)^2 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} B_1^2 & 0 \\ 0 & B_2^2 \end{pmatrix}$$

Suffice to consider the case where A is of Jordan form.

Suffice to consider one single Jordan block:

Jordan block size	J	J^n	e^{Jt}
1	(λ)	λ^n	$e^{\lambda t}$
2	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.	$\begin{pmatrix} \lambda^n & n\cdot\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$,	$e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$= \lambda\cdot I_2 + N_2$$

↑ regular nilpotent matrix of size 2.

we know $I\cdot N = N\cdot I. = N.$

$$(N_2)^2 = 0.$$

$$(\lambda I + N)^2 = \overset{\swarrow \text{ omitting } I_2.}{(\lambda + N)^2} = \lambda^2 + 2\lambda N + N^2$$
$$= \lambda^2 + 2\lambda N$$
$$= \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} + 2\lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$(\lambda I + N)^k = (\lambda + N)^k = \lambda^k + \binom{k}{1} \cdot \lambda^{k-1} \cdot N + \underbrace{\binom{k}{2} \lambda^{k-2} \cdot N^2 + \cdots}_{=0}$$

$\because N^2 = 0$
$N^3 = 0$
$\vdots$

$$= \begin{pmatrix} \lambda^k & k \cdot \lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$$

$$e^{(\lambda + N)t} = I + (\lambda + N)t + (\lambda + N)^2 \cdot \frac{t^2}{2} + \cdots\cdots$$

$$e^{(\lambda t \cdot I) + (t \cdot N)} = e^{\lambda t \cdot I} \cdot e^{tN}.$$

If A, B are 2 square matrices, and if $AB = BA$,

then $e^{A+B} = e^A \cdot e^B$

$\because$ if x, y are numbers,

$$e^x \cdot e^y = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \cdots\right)$$
$$= 1 + (x+y) + \frac{1}{2!}(x^2 + y^2 + 2xy) + \cdots\cdots$$
$$= e^{x+y}$$

the only property we use, is $xy = yx$.
Hence the same argument works for two commuting matrices. $A \cdot B$.

$$e^{tN} = I + tN + \underbrace{\frac{t^2}{2}N^2 + \cdots}_{=0} = I + tN = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$e^{(\lambda+N)t} = e^{\lambda t} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

For $\overbrace{J}^{\text{size } m} = \lambda I_m + N_m = \left.\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}\right\} m$

$$J^k = \begin{pmatrix} \lambda^k & & 0 \\ & \ddots & \\ 0 & & \lambda^k \end{pmatrix} + k \cdot \lambda^{k-1} \cdot N + \binom{k}{2} \cdot \lambda^{k-2} \cdot N^2 + \cdots$$

$$N_4 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad (N_4)^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}, \quad N_4^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

$$, \quad N_4^4 = 0.$$

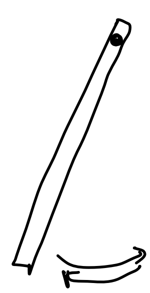
$$= \begin{pmatrix} \lambda^k & k \cdot \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} & \cdots \\ & \lambda^k & k \lambda^{k-1} & \binom{k}{2} \lambda^{k-2} \\ & & \ddots & \\ 0 & & & \lambda^k \end{pmatrix}$$

[ODE]. 3.6.2

<u>Ex</u>: (damped pendulum).

<u>Eq</u>

$$\ddot{x} = \underbrace{-kx}_{\text{Hook's Law.}} \underbrace{- c \cdot \dot{x}}_{\text{resistance}} \qquad k, c > 0.$$



- First, we turn this to a 1st order diff eq.

let $x_1(t) = x(t)$, then

$x_2(t) = \dot{x}_1(t)$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \ddot{x}_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ -k x_1 - c \cdot x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\boxed{\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

$$\Rightarrow \quad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \underbrace{e^{\begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix} t}}_{\substack{\text{evolution matrix} \\ \text{or propagator.}}} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}$$

Diagonalize A, by

$A = C \cdot J \cdot C^{-1}$.

$$0 = \det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 \\ k & \lambda + c \end{pmatrix} = \lambda(\lambda + c) + k = \underbrace{\lambda^2 + c\lambda + k}.$$

$$\boxed{\ddot{x} + c \cdot \dot{x} + k x = 0}$$

try $x = e^{\lambda t}$, get

Solve $\lambda^2 + c\lambda + k = 0$, $\quad \lambda^2 + c \cdot \lambda + \left(\frac{c}{2}\right)^2 = \left(\frac{c}{2}\right)^2 - k$

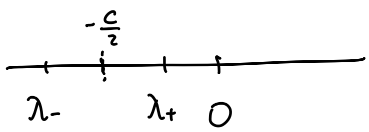
$$\left(\lambda + \frac{c}{2}\right)^2 = \left(\frac{c}{2}\right)^2 - k$$

$$\lambda + \frac{c}{2} = \pm \sqrt{\left(\frac{c}{2}\right)^2 - k}$$

$$\lambda_{\pm} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - k}.$$

<u>case 1</u> : $\lambda_{\pm}$ are distinct real sol'n.

$\Leftrightarrow \quad k < \left(\frac{c}{2}\right)^2$ (intuitively this means friction is large)



$$A = C \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} C^{-1}$$

$$e^{At} = C \cdot \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} C^{-1}$$

$$\vec{x}(t) = C \cdot \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} C^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}.$$

<u>case 2</u> : $\lambda_{\pm}$ ~~is~~ are a pair of complex conjugate solutn

$\Leftrightarrow \quad k > \left(\frac{c}{2}\right)^2.$

$$\lambda_{\pm} = -\rho \pm i\omega \qquad \omega = \sqrt{k - \left(\frac{c}{2}\right)^2}$$

$$e^{(-\rho + i\omega)t} = e^{-\rho t}\left[e^{i\omega t}\right]$$

$$= e^{-\rho t}\left[\cos(\omega t) + i \cdot \sin(\omega t)\right].$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} C^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\rightsquigarrow \quad x(t) = x_1(t) = \overset{\in \mathbb{R}}{C_1} \underbrace{e^{-\rho t} \sin(\omega t)} + \overset{\in \mathbb{R}}{C_2} \cdot \underbrace{e^{-\rho t} \cos(\omega t)}$$

$$= \alpha\, e^{\lambda_+ t} + \bar{\alpha} \cdot e^{\lambda_- t}$$

case 3 : $\lambda_+ = \lambda_- = -\frac{c}{2}$. $k = \left(\frac{c}{2}\right)^2$.

$$\cancel{J \overset{?}{=} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}} \quad \text{or} \quad J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\Rightarrow C \cdot J \cdot C^{-1} = J \neq A = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}$$

$$A = C \cdot J \cdot C^{-1}$$

$$e^{Jt} = e^{[\lambda + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)]t} = e^{\lambda t} \cdot e^{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)t} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C \cdot e^{Jt} \cdot C^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$x_1(t) = \underline{c_1} \cdot e^{\lambda t} + \underline{c_2} \cdot e^{\lambda t} \cdot t.$$

$$\dddot{x} + a_1 \ddot{x} + a_2 \cdot \dot{x} + a_3 \cdot x = 0$$

introduce new functions
$$\begin{cases} x_1(t) = x(t) \\ x_2(t) = \dot{x}_1(t) = \dot{x}(t) \\ x_3(t) = \dot{x}_2(t) = \ddot{x}(t) \end{cases}$$

$$\cancel{x_4(t) = \dot{x}_3(t) = \dddot{x}(t) = \cdots}$$

$$\dot{x}_3 = \dddot{x}(t) = -a_1 \ddot{x} - a_2 \dot{x} - a_3 x$$
$$= -a_1 x_3 - a_2 \cdot x_2 - a_3 x_1$$

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ -a_1 x_3 - a_2 x_2 - a_3 x_1 \end{pmatrix} = \underline{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\boxed{3.6.3}$ higher order

$$\det(\lambda - A) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_r)^{m_r}$$

$x(t)$ is a linear combination of

e

$$J = \begin{pmatrix} \boxed{\begin{smallmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{smallmatrix}} & \\ & \boxed{\begin{smallmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{smallmatrix}} \end{pmatrix}$$