Last time:

constant coeff diff eq:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$n \times n$, constant

in short $\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$.

the sol'n $\vec{x}(t) = e^{At} \cdot \vec{x}(0)$

a constant column vector

where exponential of a matrix is understood by its Taylor expansion: $e^M = 1 + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$

1. Boundary Condition / Initial condition.
2. Inhomogeneous Term.

1. $\frac{d}{dt}\vec{x} = A \cdot \vec{x}$ $\qquad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

the sol'n space is an n-dimensional vector space.

$$\vec{x}(t) = \underbrace{e^{At}}_{\substack{n\times n \\ \text{matrix}}} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

c_i are arbitrary constants.

we can obtain an isomorphism between $\mathbb{C}^n$ and the sol'n space.

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \mapsto e^{At} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

i.e. we can use the column vectors in e^{At} as basis of sol'n space.

To pin down the sol'n, we need to impose more constraints.

Ex: we may require that, at $t = t_0$, the sol'n $\vec{x}(t_0) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

This can be satisfied, by choosing $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$,

We plug in the general sol'n to the constraints, we get

$$e^{A \cdot t_0} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \Leftrightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = e^{-At_0} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

[recall that, if A, B commute $AB = BA$, then

$$e^{A+B} = e^{A} \cdot e^{B} = e^{B} \cdot e^{A}.$$

$$e^{A} \cdot e^{-A} = e^{0} = I.$$
]

Ex 2: Eq:

$$\begin{cases} \ddot{x}(t) = 0 \\ x(0) = 1 \\ x(1) = 2. \end{cases}$$

gen sol'n $x(t) = a + bt$. a, b are free param.

plug in the gen sol'n to the constraint, we get

$$\begin{cases} a + b\cdot 0 = 1 \\ a + b\cdot 1 = 2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 1 \end{cases}$$

<u>Ex3</u> :

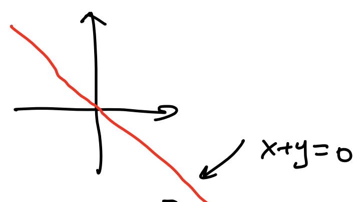
$$\begin{cases} \ddot{x}(t) = -x(t). \\ x(0) = 0 \\ x(2\pi) = 1. \end{cases} \Rightarrow x(t) = a\cdot \sin t + b\cdot \cos t.$$

$$\begin{cases} a\cdot \sin(0) + b\cdot \cos(0) = 0 \\ a\cdot \sin(2\pi) + b\cos(2\pi) = 1 \end{cases} \Leftrightarrow \begin{cases} b = 0 \\ b = 1. \end{cases}$$

no sol'n.

Inhomogeneous Equation :

<u>Ex</u> : $x + y = 0$ $\overset{\text{sol'n}}{\rightsquigarrow}$

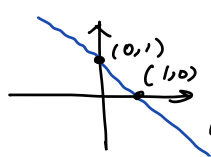


homogeneous eq.

sol'n is a vector space.

inhomogeneous eq.

<u>Ex</u> : $x + y = 1$ $\overset{\text{sol'n}}{\rightsquigarrow}$



sol'n is no longer a vector

space.

For example. $(1,0)$ and $(0,1)$ are sol'n.

but $(1,0)+(0,1) = (1,1)$ is not a sol'n.

- If (x_0, y_0) is a sol'n to $x+y=1$.

  and (x_1, y_1) is a sol'n to $x+y=0$

then

$$(x_0+x_1, y_0+y_1) \text{ is a sol'n to } x+y=1.$$

$$\begin{array}{l} x_0+y_0=1 \\ x_1+y_1=0 \end{array} \Rightarrow (x_0+x_1)+(y_0+y_1) = 1+0 = 1.$$

In general: an inhomogeneous equations is of the form

$$A\vec{x} = \vec{b}.$$

- there exists a sol'n if $\vec{b}$ is in the range of A
- If $\vec{x}_0$ satisfies $A\vec{x}=b$,

  and $\vec{x}_1$ satisfies $A\vec{x}=0$,

  then $\vec{x}_0+\vec{x}_1$ satisfies $A\vec{x}=b$.

  sol'n $\{A\vec{x}+b\}$, if not empty, then is an affine space modeled on the vector space

  sol'n. $\{A\vec{x}=0\}$

Differential Eq:

$$\frac{d}{dt} x(t) = \lambda \cdot x(t) + g(t)$$

a given function.

Let's rewrite the equation., introduce "D" $\frac{d}{dt}$.

$$(D - \lambda) \cdot x(t) = g(t). \qquad (*)$$

a particular so

If there exists a sol'n $x_0(t)$ for eq (*), then there exist many sol'n., by adding to $x_0(t)$ a sol'n of $(D-\lambda) x(t) = 0$.

general sol'n $\quad x(t) = x_0(t) + c \cdot e^{\lambda t}$.

c is free.

<u>General strategy</u>: to solve inhomog eq.

- Find a "particular sol'n" to the eq.
- Find a general sol'n to the homoge eq.
- Add them up, to get the gen sol'n to the inhom. eq.

$D = \frac{d}{dt}$. $\qquad x(t)$,

Ex : • $D \cdot x(t) = c$ (*)

$\Rightarrow x_0(t) = c \cdot t$ is a particular sol'n.

homogeneous version

$D \cdot x(t) = 0$ (**)

$\Rightarrow x_1(t) = c_1$ is a gen sol'n to

$x(t) = x_0(t) + x_1(t) = c_1 + ct$. is a gen sol'n to (*).

• $D \cdot x(t) = e^{\lambda t}$.

→ particular sol'n $x_0(t) = \frac{1}{\lambda} e^{\lambda t}$.

→ gen sol'n to the homog eq $D \cdot x(t) = 0$

$x_1(t) = c_0$.

→ gen sol'n to the inhom eq.

$x_0(t) + x_1(t) = c_0 + \frac{1}{\lambda} e^{\lambda t}$.

Ex • $(D - \lambda) \cdot x(t) = g(t)$.

• suppose $g(t) = e^{\lambda_0 t} (\underbrace{b_0 + b_1 t + \cdots + b_n t^n}_{b(t)})$.

and $\lambda_0 \neq \lambda$.

• we only find the particular sol'n here.

we first set

$$x(t) = e^{\lambda t} \cdot u(t).$$

then $\quad (D-\lambda)\cdot(e^{\lambda t}\cdot u(t)) = e^{\lambda t}\cdot D\cdot u(t)$

$$e^{\lambda t}\cdot D\cdot u(t) = e^{\lambda_0 \cdot t}\cdot b(t).$$

$$D\cdot u(t) = e^{(\lambda_0-\lambda)t}\cdot b(t).$$

$$u(t) = \int^t e^{(\lambda_0-\lambda)\tilde{t}}\cdot b(\tilde{t})\, d\tilde{t}.$$

$$x(t) = e^{\lambda t} \underbrace{\int^t e^{(\lambda_0-\lambda)\tilde{t}}\cdot b(\tilde{t})\, d\tilde{t}}$$

$$= e^{\lambda_0 t}\cdot P_n(t).$$

$\boxed{P_n(t) \text{ \& } b(t) \text{ of same deg.}}$

$$\int e^{\lambda t}\, dt = \frac{1}{\lambda} e^{\lambda t} + C.$$

$$\int e^{\lambda t}\cdot t\cdot dt = \int t\cdot d\left(\frac{e^{\lambda t}}{\lambda}\right) = t\cdot\frac{e^{\lambda t}}{\lambda} - \int \frac{e^{\lambda t}}{\lambda} dt$$

$$= e^{\lambda t}\cdot\frac{t}{\lambda} - e^{\lambda t}\cdot\frac{1}{\lambda^2}.$$

$$= e^{\lambda t}\left(\frac{t}{\lambda} - \frac{1}{\lambda^2}\right)$$

$$\left[e^{\lambda t}\frac{t}{\lambda} - e^{\lambda t}\frac{1}{\lambda^2}\right]' = \lambda\cdot e^{\lambda t}\cdot\frac{t}{\lambda} + e^{\lambda t}\cdot\frac{1}{\lambda}$$

$$- \lambda e^{\lambda t}\cdot\frac{1}{\lambda^2}$$

$$= e^{\lambda t}\cdot t.$$

polynomial of deg n.

In general, $\quad \int e^{\lambda t}\cdot t^n\cdot dt = e^{\lambda t}\cdot P_n(t)$

If $\lambda = \lambda_0$ above.

polynomial.

$$(D - \lambda)\, x(t) = e^{\lambda t} \cdot b(t)$$

then. setting $x(t) = e^{\lambda t} \cdot u(t)$,

we have

$$D \cdot u(t) = b(t).$$

$$u(t) = \int b(t)\, dt$$

$$= P(t)$$

polynomial of deg $(n+1)$.

$$x(t) = e^{\lambda t} \cdot P(t).$$

particular sol'n.

See Givental ODE book Ex 3.7.3 (b) (c).