Today :

1. Fourier series
2. some important PDEs.
3. Sol'n using Fourier series.

① Consider the interval $[-L, L]$.
consider complex valued ( continuous, or Riemann integrable) function on $[-L, L]$.

Let $f(x)$, $g(x)$ be such nice functions on $[-L, L]$.
we define " inner product"

$$\langle f, g \rangle = \frac{1}{2L} \int_{-L}^{L} \overline{f(x)} \cdot g(x) \, dx$$

[ compare this with Hermitian inner product on $\mathbb{C}^N$.
$\vec{x} = (x_1, \cdots, x_N)$ , $\vec{y} = (y_1, \cdots, y_N)$ $\in \mathbb{C}^N$

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{N} \overline{x_i} \cdot y_i$$ ]

- $e_n(x) := e^{\frac{\pi i x}{L} \cdot n}$ complex valued. , $n \in \mathbb{Z}$.

[ it is $2L$-periodic, $e_n(x+2L) = e_n(x)$.

$$e_n(x+2L) = e^{\frac{\pi i}{L}(x+2L)n} = e^{\frac{\pi i}{L} x \cdot n + 2\pi i \cdot n}$$

$$= e^{\frac{\pi i}{L} \cdot x \cdot n} \cdot \underbrace{e^{2\pi i n}}_{=1}$$ ]

$$\langle e_n(x), e_m(x) \rangle = \frac{1}{2L} \int_{-L}^{L} \overline{e^{\frac{\pi i}{L} x \cdot n}} \cdot e^{\frac{\pi i}{L} x \cdot m} \cdot dx$$

$$= \frac{1}{2L} \int_{-L}^{L} e^{\frac{\pi i}{L} x(m-n)} \cdot dx.$$

$$= \begin{cases} 1 & m = n. \\ 0 & m \neq n. \end{cases}$$

$k = m - n \neq 0$

$$\int_{-L}^{L} e^{\frac{\pi i}{L} \cdot k \cdot x} dx = \left. \frac{e^{\frac{\pi i}{L} \cdot k \cdot x}}{\frac{\pi i}{L} \cdot k} \right|_{-L}^{L}$$

$$= \frac{e^{\frac{\pi i}{L} \cdot L \cdot k} - e^{\frac{\pi i}{L}(-L) k}}{\frac{\pi i}{L} \cdot k} = \frac{e^{k\pi i} - e^{-k\pi i}}{\frac{\pi i}{L} \cdot k}$$

$$= \frac{(-1)^k - (-1)^{-k}}{\frac{\pi i}{L} \cdot k} = 0. \qquad e^{\pi i} = (-1)$$

<u>remark</u> : if we consider the " $L^2$ space on $[-L, L]$ " with the above inner product, then $\{e_n(x)\}_{n \in \mathbb{Z}}$. forms an orthonormal basis.

- Given any continuous function $f(x)$ on $[-L, L]$, we can define its complex Fourier coefficients. $\{c_n\}_{n \in \mathbb{Z}}$.

$$c_n := \langle e_n, f \rangle = \frac{1}{2L} \int_{-L}^{L} e^{-\frac{\pi i}{L} \cdot x \cdot n} \cdot f(x)\, dx.$$

complex Fourier series :

$$f(x) \overset{?}{=} \sum_{n \in \mathbb{Z}} C_n \cdot e_n(x).$$

compare : suppose $V$ is a finite dim, Hermitian vectorspace

and $\{e_1, \dots, e_n\}$ is an ONB.

Then, for any $v \in V$, we have

$$v = \sum_{i=1}^{n} c_i \cdot e_i \quad , \quad \text{and the coeff}$$

$$c_i = \langle e_i, v \rangle.$$

- For $f(x)$ continuous, then for any $x \in [-L, L]$, we have

$$f(x) = \sum_{n \in \mathbb{Z}} e_n(x) \cdot \langle e_n, f \rangle = \lim_{N \to \infty} \sum_{n=-N}^{N} e_n(x) \langle e_n, f \rangle.$$

- Real functions's Fourier expansion.

Given a real valued function $f(x)$ on $[-L, L]$, we can use

$$1, \quad \sin\left(\frac{\pi x}{L} \cdot n\right), \quad \cos\left(\frac{\pi x}{L} \cdot n\right) \quad , \quad n = 1, 2, \dots$$

to expand $f$. i.e,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cdot \cos\left(\frac{\pi x}{L} \cdot n\right) + b_n \cdot \sin\left(\frac{\pi x}{L} \cdot n\right)\right).$$

These $\overset{S_n(x)}{\overset{\shortparallel}{\sin\left(\frac{\pi x}{L}\cdot n\right)}}$ and $\overset{C_n(x)}{\overset{\shortparallel}{\cos\left(\frac{\pi x}{L}\cdot n\right)}}$ 1 forms an ONB.

$$e_n(x) = e^{\frac{\pi i}{L}x\cdot n} = \cos\left(\frac{\pi}{L}\cdot x\cdot n\right) + i\sin\left(\frac{\pi x}{L}\cdot n\right)$$
$$= C_n(x) + i\cdot S_n(x).$$

$$e_{-n}(x) = C_{-n}(x) + i\,S_{-n}(x)$$
$$= C_n(x) - i\,S_n(x).$$

$$C_n(x) = \frac{e_n + e_{-n}}{2}$$

$$S_n(x) = \frac{e_n - e_{-n}}{2i}$$

$$\langle C_n, S_n\rangle = \left\langle \frac{e_n + e_{-n}}{2}, \frac{e_n - e_{-n}}{2i}\right\rangle$$
$$= \frac{1}{4i}\Big(\langle e_n, e_n\rangle - \langle e_{-n}, e_{-n}\rangle\Big)$$
$$= \frac{1}{4i}(1-1) = 0.$$

$$\langle C_n, C_n\rangle = \left\langle \frac{e_n + e_{-n}}{2}, \frac{e_n + e_{-n}}{2}\right\rangle$$
$$= \frac{1}{4}\cdot\left(\langle e_n, e_n\rangle + \langle e_{-n}, e_{-n}\rangle\right)$$
$$= \frac{1}{4}\cdot 2 = \frac{1}{2}.$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cdot C_n(x) + b_n \cdot S_n(x) \right)$$

- To extract $a_0$, we do.

$$\langle 1, f(x) \rangle = \langle 1, a_0 \rangle = \frac{1}{2L} \int_{-L}^{L} 1 \cdot a_0 \, dx$$

$$\frac{1}{2L} \int_{-L}^{L} f(x)\, dx = a_0.$$

$$\Rightarrow \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx.$$

$C_n(x) = \cos\left(\frac{\pi x}{L} \cdot n\right).$

- to extract $a_n$, we apply $\langle C_n, - \rangle.$

$$\langle C_n, f(x) \rangle = a_n \langle C_n, C_n \rangle = \frac{1}{2} a_n.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} \cos\left(\frac{\pi x}{L} \cdot n\right) \cdot f(x) \cdot dx.$$

- $$b_n = \frac{1}{L} \int_{-L}^{L} \sin\left(\frac{\pi x}{L} n\right) \cdot f(x)\, dx.$$

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Ex: let $f(x) = x$ on $[-L, L]$.

find its complex Fourier coefficients.

& real ______________.

(read [ODE]).

② Famous $\overset{\text{partial}}{\text{Diff}}$ Eq.

Given a function that depends on more than one variable. say $f(x,y)$,

we can talk about its partial derivative.

$$\partial_x f(x_0, y_0) := \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon, y_0) - f(x_0, y_0)}{\varepsilon}.$$

$$\partial_y f(x_0, y_0) := \lim_{\varepsilon \to 0} \frac{f(x_0, y_0+\varepsilon) - f(x_0, y_0)}{\varepsilon}.$$

we can definer higher order partial derivatives

$$\partial_x(\partial_x f), \qquad \partial_y(\partial_x f),$$

$$\partial_x(\partial_y f), \qquad \partial_y(\partial_y f).$$

In the case that $f$ is "twice differentiable"

then. $\partial_x \partial_y f = \partial_y \partial_x f.$

Ex: $f(x,y) = x \cdot y.$ then

$$\partial_x f = y, \qquad \partial_y f = x.$$

$$\partial_x \partial_x f = \partial_x (y) = 0, \quad \partial_y \partial_y f = 0.$$

$$\partial_x \partial_y f = \partial_x (x) = 1.$$

$$\partial_y \partial_x f = \partial_y (y) = 1.$$

- We will use notation. $f_x = \partial_x f$, $f_{xx} = \partial_x \partial_x f = \partial_x^2 f$.

- <u>Laplacian</u> : $\Delta$.

on $\mathbb{R}^n$ $\quad \Delta = (\partial_{x_1})^2 + (\partial_{x_2})^2 + \cdots + (\partial_{x_n})^2.$

- $\Delta f$ measures how much a function "<u>bend</u>"

<u>Ex</u>: on $\mathbb{R}$. $\quad \Delta f(x) = \partial_x^2 f(x)$.

$$\Delta f(x) = 0 \iff \partial_x^2 f(x) = 0$$
$$\iff f(x) = a + bx.$$

<u>straight line</u>.

<u>Ex</u>: on $\mathbb{R}^2$, $\quad \Delta f(x,y) = \partial_x^2 f + \partial_y^2 f$.

$$\Delta f(x,y) = 0 \iff \partial_x^2 f(x,y) + \partial_y^2 f(x,y) = 0$$

<u>ex</u> sol'n: $f(x,y) = a + bx + cy$.

$f(x,y) = xy$

or $f(x,y) = x^2 - y^2$,

......

The sol'n to $\Delta f(x,y) = 0$ ~~is~~ are called harmonic functions.

- Eigenvalue problem for $\Delta$:

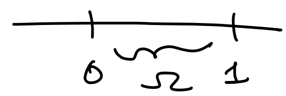
<u>Ex</u>: Let $\Omega \subset \mathbb{R}^n$, $f: \Omega \to \mathbb{R}$.

$$\begin{cases} \Delta f = \lambda f \\ f|_{\partial\Omega} = 0. \end{cases}$$

$\lambda$ : eigenvalue for $\Delta$.
$f$ : eigenvector.

<u>Ex</u>: $[0,1] \subset \mathbb{R}$. $f(x)$ s.t.

$$\begin{cases} \partial_x^2 f = \lambda f \\ f(0)=0, \quad f(1)=0 \end{cases}$$




in 1-dim, we have the sequence of sol'n. $n = 1, 2, \cdots$.

$$\begin{cases} f_n(x) = \sin(\pi x \cdot n). \\ \lambda_n = -(\pi n)^2. \end{cases}$$

$$\partial_x f_n(x) = \partial_x \sin(\pi n \cdot x) = \cos(\pi n \cdot x) \cdot (\pi n)$$

$$\partial_x^2 f_n(x) = \partial_x \cos(\pi n \cdot x) \cdot (\pi n) = -\sin(\pi n \cdot x) \cdot (\pi n)^2$$

- Heat equation: $f: \underset{t}{\mathbb{R}} \times \underset{x}{\mathbb{R}^n} \to \mathbb{R}$.

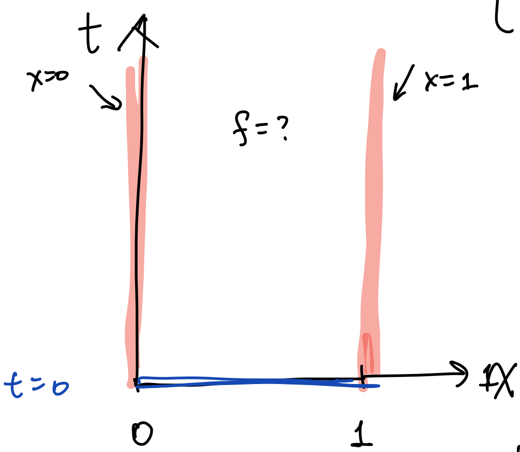
$$\partial_t f(t,x) = \Delta_x f(t,x)$$

in special case:

want $\quad f: [0,\infty)_t \times [0,1]_x \to \mathbb{R}.$

such that.

$$\begin{cases} \partial_t f(t,x) = \partial_x^2 f(t,x) \\ f(0,x) = f_0(x) \\ f(t,0) = 0, \quad f(t,1) = 0 \end{cases}$$




Fourier expansion.

If $\quad f_0(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(\pi x \cdot n).$

then we know how each $\sin(\pi x \cdot n)$ evolve.

i.e.,

if $\quad f_0(x) = \sin(\pi x \cdot n),$

then $\quad f(t,x) = e^{-(\pi n)^2 \cdot t} \cdot \sin(\pi x \cdot n).$

$$\partial_t f = -(\pi n)^2 \cdot e^{-(\pi n)^2 t} \cdot \sin(\pi x \cdot n)$$

$$\partial_{xx} f = e^{-(\pi n)^2 t} \cdot (-(\pi n)^2) \cdot \sin(\pi x \cdot n).$$

$$\Rightarrow \partial_t f = \partial_{xx} f.$$

then $\quad f(t,x) = \sum_{n=1}^{\infty} b_n \cdot e^{-(\pi n)^2 t} \cdot \sin(\pi x \cdot n).$