Diff Eq:

(1) $$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = A \cdot \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

A: constant coeff matrix.

sol'n: $\vec{x}(t) = e^{At} \cdot \vec{x}(0)$

where $e^{At} = I_n + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3$

- if $A = C^{-1} \cdot J \cdot C$, J Jordan form
  then $e^{At} = C^{-1} \cdot e^{Jt} \cdot C$
- in special case, if A is diagonalizable.
  $J = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$
  $e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}.$

- $\vec{x}(0)$ contain n free parameters, $x_1(0), \cdots, x_n(0)$.

(2) Higher order const coeff equation: (Homog).

$$\left(\frac{d}{dt}\right)^n x(t) + a_1\left(\frac{d}{dt}\right)^{n-1} x(t) + \cdots + a_{n-1}\frac{d}{dt}x(t) + a_n \cdot x(t) = 0$$

we introduce new unknown functions:

$x_0(t) = x(t)$
$x_1(t) = \frac{d}{dt}x(t)$

$\vdots$

$$x_{n-1}(t) = \left(\frac{d}{dt}\right)^{n-1} x(t)$$

These new functions satisfy

$$\left(\frac{d}{dt}\right)\begin{pmatrix} x_0 \\ \vdots \\ x_{n-1}\end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ (x(t))^{(n)}\end{pmatrix} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_{n-1}(t) \\ -a_1 x_{n-1} \cdots - a_n \cdot x_0\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 & & & \\ \vdots & 0 & 1 & & \\ \vdots & \vdots & & 1 & \\ 0 & \vdots & & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{pmatrix}}_{A}\begin{pmatrix} x_0 \\ \vdots \\ \vdots \\ x_{n-1}\end{pmatrix}.$$

this reduces to case (1).

the short way to solve this, is to find the characteristic polynomial:

try to ~~solve~~ plug in $x = e^{\lambda t}$.

get equation.

$$\underbrace{\left(\lambda^n + a_1 \cdot \lambda^{n-1} + \cdots + a_n\right)}_{= \det(\lambda - A).} \cdot e^{\lambda t} = 0$$

(3). Solving PDE of type. $\Delta_x u(t,x) = \cdots$. ($\partial_t u$, or $\partial_{tt} u$, ...)

<u>Heat</u>: $\partial_t u(t,x) = \partial_{xx} u(t,x)$.

we can use separation of variable to obtain.

"<u>basis</u>" of sol'n:

$$u(t,x) = f(t)\cdot g(x),$$

then

$$\partial_t \left(f(t)\cdot g(x)\right) = \partial_{xx}\left(f(t)\, g(x)\right).$$

$$\Longleftrightarrow \quad (\partial_t f)\cdot g = f\cdot \partial_{xx} g.$$

$$\Longleftarrow \quad \frac{\partial_t f(t)}{f(t)} = \frac{\partial_{xx} g(x)}{g(x)} = \text{const}.$$

if $f\cdot g \neq 0$

$$\Rightarrow \begin{cases} \partial_t f(t) = \lambda f(t) & \Rightarrow f(t) = e^{\lambda t} f(0). \quad \text{for some } \lambda. \\ \partial_{xx} g(x) = \lambda\cdot g(x) & \Rightarrow g(x) = e^{\sqrt{\lambda} t}\cdot c_1 + e^{-\sqrt{\lambda} t} c_2. \end{cases}$$