Math H54 Final Dec 14, 2022

8 problems, 10 points each.

(1) Decide which of the following subset is a vector space. 2 points each, no explanation needed.
  (a) The solution set $\{x : Ax = 0\}$, where $A$ is a matrix, and $x$ is a column vector.
  (b) The subset $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 0\}$
  (c) The set of $n \times n$ matrices $A$ where $\det A = 0$.
  (d) The solution to some homogeneous linear differential equation, with homogenous boundary conditions. e.g. $f_{xx} + f_t + f_{tt} = 0$, and $f(x = 0, t) = 0$.
  (e) the set of linear operators $T : \mathbb{R}^2 \to \mathbb{R}^2$, such that $T$ satisfies

$$T\begin{pmatrix}1\\0\end{pmatrix} \perp \begin{pmatrix}1\\0\end{pmatrix}$$

  where $\perp$ is with respect to the standard inner product on $\mathbb{R}^2$.

(2) Quick questions, no justification needed, 2 points each.
  (a) For what values of $c$ in $\mathbb{C}$ is the matrix $\begin{pmatrix}1 & c\\0 & 2\end{pmatrix}$ diagonalizable?
  (b) For what values of $c$ in $\mathbb{C}$ is the matrix $\begin{pmatrix}1 & c\\0 & 1\end{pmatrix}$ diagonalizable?
  (c) What is the rank of the matrix $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{pmatrix}$? (bonus: what if you change the matrix size to $n$?)
  (d) Can all degree 2 polynomials with $\mathbb{C}$ coefficients be expressed as $\mathbb{C}$-linear combinations of $x - 1, 2x - 1, 3x^2 + 1$? If you think so, express $x^2$ as a linear combination of them.
  (e) If a matrix $A$ has characteristic polynomial $p(\lambda) = (\lambda - 1)^2(\lambda - 2)$, then is it true that $A$ is not diagonalizable?

(3) Use row reduction, or whatever method you prefer, and solve the following equation

$$\begin{pmatrix}0 & 1 & 2\\2 & 0 & 1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

(4) Consider the equation for $u(x, t)$

$$u_t = u_{xx} + 2u_x, \quad t > 0, x \in (0, 1)$$

a: (5pts) If $u(x, t) = f(x)g(t)$ satisfies the equation, then what equations do $f$ and $g$ satisfy?
b: (5pts) If we further require that $u(x, t) = 0$ when $x = 0$ and $x = 1$, can you write down the general solution to the above equation?

(5) Find the fundamental solution for the equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t), \quad A = \begin{pmatrix} 9 & -4 \\ 9 & -3 \end{pmatrix}.$$

Namely, find a $2\times 2$ matrix $\Psi(t)$ explicitly, such that $\frac{d}{dt}\Psi(t) = A\Psi(t)$ and $\Psi(0) = I$. You may use the following info

$$A\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad A\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(6) (a) (5 pts) Find an order 3 homogeneous differential equation about $y(x)$, that is, an equation of the form

$$y'''(x) + a_1 y''(x) + a_2 y'(x) + a_3 y(x) = 0$$

such that $xe^x + e^{-x}$ is a solution.

(b) (5 pts) Find a particular solution to

$$y'(x) + y(x) = e^{-x}.$$

(7) Consider the function $f(x) = |\sin(x)|$ for $x \in [-\pi, \pi]$. Find its Fourier series on $(-\pi, \pi)$ in the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mx) + b_m \sin(mx).$$

Namely, compute all $a_m, b_m$. You may need the formula that

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

(8) Put the following matrix in Jordan normal form,

$$A = \begin{pmatrix} 1 & c & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

that is, find invertible $C$, such that $A = CJC^{-1}$ where $J$ Jordan form.

(For full credits, do the case where $c = 1$. For half credits (5 points), do the case where $c = 0$.)