Problem 1

Consider the vector $(5, 3)$, expressed in the standard basis (i.e. $\{(1, 0), (0, 1)\}$). What are its coordinates with respect to the basis defined by $B = \{(1, 1), (1, -1)\}$?

Solution

There are two appropriate methods to do this:

1. Write the coordinates of the new vector as (a, b) and solve the system of linear equations one gets by setting $(5, 3) = a(1, 1) + b(1, -1)$. In this case, one has $a + b = 5$, $a - b = 3$, from we get $(a, b) = (4, 1)$.

2. The other method is recognize that the basis vector given are orthogonal, but not orthonormal. So, we can normalize them to have length 1 and use the result discussed at the end of the first lecture (For an orthonormal basis $\{e_1, e_2\}$, we have $v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2$). We normalize to get the basis $B' = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$. The coordinates with respect to this orthonormal basis are

$$(5, 3) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{8}{\sqrt{2}} \text{ and } (5, 3) \cdot (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{2}{\sqrt{2}}.$$

   Since the basis B' is composed of the same vectors as B scaled down by a factor of $\sqrt{2}$, when we switch back to B we must scale down the coordinates. We get that the new coordinates of $(5, 3)$ are $(\frac{8}{\sqrt{2}\sqrt{2}}, \frac{2}{\sqrt{2}\sqrt{2}}) = (4, 1)$.

Problem 2

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is linear if $f(cx) = cf(x)$, and $f(x + y) = f(x) + f(y)$. Are the following functions linear:

a) $f(x) = 2x$,

b) $f(x) = x^2$,

c) $f(x) = x + 2$?

Solution

1. Yes. $f(cx) = 2(cx) = c \cdot 2x = cf(x)$ and $f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y)$.

2. No. $f(cx) = (cx)^2 = c^2x^2 \neq cf(x) = cx^2$ unless $c = 0, 1$, but this must hold for all c.

3. No. $f(cx) = cx + 2 \neq cf(x) = c(x + 2)$ unless $c = 1$. Again, this must hold for all c. Indeed, $f(x + y) = x + y + 2 \neq f(x) + f(y) = x + y + 4$, so neither condition holds.

Problem 3

Let ABC be a triangle. Is it true that there exists only one point M, such that $MA + MB + MC = 0$? What if we change the condition to $2MA + MB + MC = 0$?

Solution

We wish to show that there exists a unique solution to $MA+MB+MC = 0$. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ be fixed, and let $M = (m_1, m_2)$ be unknown. We can treat the x and y coordinates independently (but since they are symmetric we can do both at the same time):

$$(m_i - a_i) + (m_i - b_i) + (m_i - c_i) = 0$$

hence

$$m_i = \frac{1}{3}(a_i + b_i + c_i)$$

is the unique solution. So, there exists a point $M = (m_1, m_2) = \frac{1}{3}(a_1 + b_1 + c_1, a_2 + b_2 + c_2)$, and it is unique since the equation has only one solution. Incidentally, this also proves that $M = \frac{1}{3}(A + B + C)$, which we can recognize as the barycenter of triangle ABC.

For the second problem, there are two approaches, the first is to reproduce a similar calculation as was done above. We'll prove it this way first: Setting the coordinates as before, we have

$$2(m_i - a_i) + (m_i - b_i) + (m_i - c_i) = 0$$

hence

$$m_i = \frac{2a_i + b_i + c_i}{4}.$$

Again, a solution to the equation exists and is evidently unique (we shall make this notion of 'evidently unique' more precise in the future).

The second is to consider the triangle $A'BC$, where A' is the point along the median originating at A which is equidistant from M and the line BC (i.e. the same triangle as before but 'push' the vertex A in a little closer). Then, by the previous part, we know that $MA + MB + MC = 0$ has a unique solution for M, but by the definition of A', this is precisely the equation $2MA' + MB + MC = 0$, so this equation also has a unique solution.