Problem 1

Find a linear transformation which maps the unit circle to an ellipse with major and minor axes of lengths 5 and 3, respectively. The major axis should align with the y-axis.

Solution

We wish to map the unit circle

$$x^2 + y^2 = 1$$

to the ellipse

$$\frac{x'^2}{9} + \frac{y'^2}{25} = 1,$$

which can equivalently be expressed as

$$\left(\frac{x'}{3}\right)^2 + \left(\frac{y'}{5}\right)^2 = 1.$$

So, the transformation needed is $x = \frac{x'}{3}$ and $y = \frac{y'}{5}$. In terms of the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, this is represented by the linear transformation

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix},$$

since $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 5y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

Problem 2

Is there a linear transformation that takes the three points (1,0), (1,1) and (0,1) to (2,0), (1,1), (0,2)?

Solution

No such transformation exists. We proceed by contradiction. Suppose such a transformation T existed. Then,

$$T(1,0) = (2,0)$$

and

$$T(0,1) = (0,2).$$

Then, by linearity we must have

$$T(1,1) = T((1,0) + (0,1)) = (2,0) + (0,2) = (2,2),$$

but $(2,2) \neq (1,1)$, so no such transformation exists.

Problem 3

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Compute A^2, A^{-1} and $A + B$ where

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1+1) & (1-1) \\ (1-1) & (1-(-1)) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

For A^{-1} we can apply the formula for the inverse of a 2×2 matrix, or we can notice that since $A^2 = 2\mathbb{I}$, then $A \cdot \frac{1}{2}A = \mathbb{I}$ and hence $\frac{1}{2}A = A^{-1}$; i.e.

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$$