Problem 1

Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that the area of the parallelogram defined by $T(0,1)$ and $T(1,0)$ is $\frac{1}{2}$.

Bonus: (5pts) Using your solution to the first part or otherwise, find a family of linear transformations with this property.

Solution

There are many possible solutions here, the most straightforward of these is

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Then, $T(1,0) = (1,0)$ and $T(0,1) = (0, \frac{1}{2})$. The parallelogram defined by $(1,0)$ and $(0, \frac{1}{2})$ is just a rectangle with base 1 and height $\frac{1}{2}$. Hence, its area is $\frac{1}{2}$.

Bonus: In general, any matrix with determinant $\pm\frac{1}{2}$ will make the area of the resulting parallelogram equal to $\frac{1}{2}$. If you used this fact but didn't prove it, then you were given only 3 bonus points. Full points were awarded for constructing a family of transformations for which it is self-evident that each member has this property or it was proven that each member has the property. Some examples of families for which it is self-evident are

$$T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad T_n = \begin{bmatrix} n & 0 \\ 0 & \frac{1}{2n} \end{bmatrix}$$

In the first case, T_θ maps the standard basis vectors to the rectangle above, then simply rotates the rectangle, obviously preserving the area. In the second case, the parallelogram is a rectangle with width n and height $\frac{1}{2n}$.

Problem 2

Compute the determinant of the following $(n+m) \times (n+m)$ matrix

$$M = \begin{bmatrix} 0 & A \\ B & C \end{bmatrix},$$

where A is an $n \times n$ matrix and B is $m \times m$.

Solution

We use the fact that the determinant of an upper-triangle block matrix is the product of the determinants of the diagonal block matrices. Note, however, that the zero matrix in the upper-left and C in the bottom-right are not square matrices and hence do not have a determinant. Only A and B have a determinant. Hence, we have to swap rows/columns so that A and B lie on the diagonal. Note that if move the bottom m rows to the top, then the resulting matrix is

$$M' = \begin{bmatrix} B & C \\ 0 & A \end{bmatrix},$$

which has determinant $\det(B)\det(A)$. So,

$$\det(M) = (-1)^k \det(B)\det(A),$$

where k is the number of row swaps needed to get to M'. Denote the rows of M by $r_1, \dots, r_n, r_{n+1}, \dots, r_{n+m}$. For $i \geq n+1$ takes n transpositions to move r_i into position r_{i-n} by doing n 'upwards' transpositions (swapping with the row directly above). Applying these n transpositions for each of the m bottom rows of M yields the matrix M'. Hence,

$$\det(M) = (-1)^{nm} \det(B)\det(A).$$

Problem 3

Compute the determinant of the following matrix

$$A_n = \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & -1 \\ a_n & a_{n-1} & \dots & a_2 & \lambda + a_1 \end{bmatrix}$$

Hint: Use induction.

Solution

Following the hint, we consider induction on n. Computing the determinant for $n = 1, 2, 3$, we observe the pattern (defining $a_0 = 1$)

$$\det(A_n) = \sum_{i=0}^{n} \lambda^{n-i} a_i.$$

Base Case $n = 1$: In this case, A_1 is the 1×1 matrix $\lambda + a_1$, so the determinant is obviously $\lambda + a_1$. This agrees with the formula $\sum_{i=0}^{1} \lambda^{1-i} a_i = \lambda + a_1$.

Inductive Step: Assume that the formula holds for some $n-1$, we show it holds for n. We compute the determinant using cofactor expansion along the first column. So,

$$\det(A_n) = \lambda \det(A_{n-1}) + (-1)^{n-1} a_n \det \begin{bmatrix} -1 & 0 & \dots & 0 \\ \lambda & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \lambda & -1 \end{bmatrix}$$

$$= \lambda \left(\sum_{i=0}^{n-1} \lambda^{n-1-i} a_i \right) + a_n$$

$$= \sum_{i=0}^{n-1} \lambda^{n-i} a_i + a_n$$

$$= \sum_{i=0}^{n} \lambda^{n-i} a_i,$$

where the second equality follows by the induction hypothesis and from the fact that

$$\begin{bmatrix} -1 & 0 & \dots & 0 \\ \lambda & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \lambda & -1 \end{bmatrix}$$

is a lower-triangular matrix of size $(n-1) \times (n-1)$ with -1's along the diagonal, so its determinant is $(-1)^{n-1}$.