Problem 1

Prove the Rank-Nullity Theorem: If U and V are vector spaces and $T : U \to V$, then

$$\dim(V) = \dim(\text{im}(T)) + \dim(\ker(T)) = rk(T) + \mathcal{N}(T).$$

You may use the Rank Theorem: A linear map $T : U \to V$ of rank r between two vector spaces of dimensions n and m is given by the matrix $E_r = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}$ in suitable bases of U and V.

Solution

Problem 2

Let U and V be finite dimensional vector spaces over a scalar field $\mathbb{K}$. Prove that if $\dim(U) > \dim(V)$, then any linear transformation $T : U \to V$ is not injective.

Solution

Problem 3

Let $T : \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear transformation. Prove the following

a The nullity of T is $n - 1$.

b If $B = \{v_1, \ldots, v_{n-1}\}$ is a basis for $\mathcal{N}(T)$ and $w \notin \mathcal{N}(T)$, then $B' = \{v_1, \ldots, v_{n-1}, w\}$ is a basis of $\mathbb{R}^n$.

c Each vector $u \in \mathbb{R}^n$ can be expressed as

$$u = v + \frac{T(u)}{T(w)} w$$

for some $v \in \mathcal{N}(T)$.

Solution