Problem 1

Prove the Rank-Nullity Theorem: If U and V are vector spaces and $T : U \to V$, then

$$\dim(U) = \dim(\text{im}(T)) + \dim(\ker(T)) = rk(T) + \mathcal{N}(T).$$

You may use the Rank Theorem: A linear map $T : U \to V$ of rank r between two vector spaces of dimensions n and m is given by the $m \times n$ matrix $E_r = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}$ in suitable bases of U and V.

Solution

Let $T : U \to V$ be a linear map of rank r and by the Rank Theorem, pick suitable bases of U, V such that the matrix of T in those bases is given by E_r. Recall that $\ker(T) = \{x \in U : Tx = 0\}$. Note that any vector $v \in U$ which has its first r components equal to zero will map to $0 \in V$ under T. Also, any vector which is nonzero in one of the first r component is not mapped to 0 under T. Hence, the kernel of T is exactly the set of vectors which are zero in all of the first r components. A basis for the kernel of T is hence $\{e_{r+1}, e_{r+2}, \dots, e_n\}$ where e_i denotes the vector with a 1 in the i^{th} component and elsewhere. So, $\dim(\ker(T)) = n - r$. Since $\dim(U) = n$, it only remains to prove that $\dim(\text{im}(T)) = r$. This follows since $\{Te_1, Te_2, \dots, Te_r\}$ is a basis for the image of T, as $Te_{r+1}, Te_{r+2}, \dots, Te_n$ are all zero.

Problem 2

Let U and V be finite dimensional vector spaces over a scalar field $\mathbb{K}$. Prove that if $\dim(U) > \dim(V)$, then any linear transformation $T : U \to V$ is not injective.

Solution

By the Rank-Nullity Theorem,

$$\dim(U) = \dim(\text{im}(T)) + \dim(\ker(T)),$$

and note that $\dim(\text{im}(T)) \leq \dim(V)$. Hence, $\dim(\ker(T)) > 0$ so some nonzero vector $x \in U$ maps to zero under T. Hence, T is not injective.

Problem 3

Let $T : \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear transformation. Prove the following

a) The nullity of T is $n - 1$.

b) If $B = \{v_1, \dots, v_{n-1}\}$ is a basis for $\mathcal{N}(T)$ and $w \notin \mathcal{N}(T)$, then $B' = \{v_1, \dots, v_{n-1}, w\}$ is a basis of $\mathbb{R}^n$.

c) Each vector $u \in \mathbb{R}^n$ can be expressed as

$$u = v + \frac{T(u)}{T(w)} w$$

for some $v \in \mathcal{N}(T)$.

Solution

a) By the Rank-Nullity Theorem,

$$\dim(\mathbb{R}^n) = \dim(\operatorname{im}(T)) + \dim(\ker(T)).$$

Since T is nonzero, $\dim(\operatorname{im}(T)) = 1$ and obviously $\dim(\mathbb{R}^n) = n$. Hence, $\dim(\ker(T)) = n - 1$, as desired.

b) It suffices to prove that w is linearly independent from the basis B, since then B' is a set of n linearly independent vectors in $\mathbb{R}^n$, so it is a basis for $\mathbb{R}^n$. Suppose that the set B' is not a linearly independent set. Hence, there exists coefficients $c_1, \ldots, c_{n-1}, c_n$ such that

$$c_1 v_1 + \cdots + c_{n-1} v_{n-1} + c_n w = 0.$$

But then,

$$w = -\frac{1}{c_n}(c_1 v_1 + \cdots + c_{n-1} v_{n-1}),$$

so w can be expressed as a linear combination of elements of B, implying that $w \in \mathcal{N}(T)$, a contradiction.

c) Since B' is a basis for $\mathbb{R}^n$, we can express any $u \in \mathbb{R}^n$ as

$$u = c_1 v_1 + \cdots + c_{n-1} v_{n-1} + c_n w$$

for suitable coefficients $c_1, \ldots, c_n$. Denote

$$v = c_1 v_1 + \cdots + c_{n-1} v_{n-1}$$

so that

$$u = v + c_n w$$

where $v \in \mathcal{N}(T)$ since it is a linear combination of basis vectors for $\mathcal{N}(T)$. It remains to prove that $c_n = \frac{T(u)}{T(w)}$. Since $u = v + c_n w$, we have that

$$\begin{aligned} T(u) &= T(v + c_n w) \\ &= T(v) + c_n T(w) \\ &= c_n T(w), \end{aligned}$$

since $T(v) = 0$. Hence, $c_n = \frac{T(u)}{T(w)}$, as desired.