Name: ______________________________

I_n denote the identity matrix of size n. S_n is the set of permutations of n elements. For any prime p, we let $\mathbb{F}_p = \mathbb{Z}/(p)$ denote the finite field of p elements. Bonus questions do not count into points.

1. (50 pts, 10 points each) True or False, please justify your answers.
   (a) There exists a linear transformation of $\mathbb{R}^2$, that takes the conic curve $x^2 - y^2 = 1$ to $x^2 + y^2 = 1$. (bonus: what if we change $\mathbb{R}$ to $\mathbb{F}_5$? $\mathbb{F}_p$?.)
   (b) For any $\sigma_1, \sigma_2 \in S_n$, we have $l(\sigma_1 \circ \sigma_2) = l(\sigma_1) + l(\sigma_2)$.
   (c) The 3 vectors $(1,1,0), (2,3,0), (4,5,0)$ in $\mathbb{R}^3$ are linearly independent.
   (d) If A is an $n \times n$ matrix with determinant 1 and integer entries, then A^{-1} also has integer entries.
   (e) If $z \in \mathbb{C}$ satisfies that $\bar{z}^2 = 1/z^3$, then $|z| = 1$.

2. (50 points, 10 points each)
   (a)
   $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = ?$$
   (b)
   $$\det \begin{pmatrix} 0 & e & d \\ a & b & 0 \\ g & 0 & f \end{pmatrix} = ?$$
   (c) Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear map of taking the derivative,
   $$T(f(x)) = f'(x).$$
   Is T surjective? Is T injective? (bonus: what if we change $\mathbb{R}$ to $\mathbb{F}_5$? or $\mathbb{F}_p$? can you write down $\ker(T), im(T)$?)
   (d) Let $V = Map(\mathbb{R}, \mathbb{R})$ be the linear space of all functions on $\mathbb{R}$ and let $T : V \to \mathbb{R}$ be a linear map. Suppose we know that,
   $$T(\sin(x)) = 1, \quad T(\cos(x)) = 3,$$
   then for any given $\theta \in \mathbb{R}$,
   $$T(\sin(x + \theta)) = ?$$
   You may use $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.
   (e) Let p be any prime (if you wish, you can let $p = 11$), let $S = \{(x, y, z) \in \mathbb{F}_p^3 \mid 2x + 3y + 5z = 1\}$. Is S a linear subspace of $\mathbb{F}_p^3$? What is the size of S?