Name: ______________________________

I_n denote the identity matrix of size n. S_n is the set of permutations of n elements. For any prime p, we let $\mathbb{F}_p = \mathbb{Z}/(p)$ denote the finite field of p elements. Bonus questions do not count into points.

1. (50 pts, 10 points each) True or False, please justify your answers.

F (a) There exists a linear transformation of $\mathbb{R}^2$, that takes the conic curve $x^2 - y^2 = 1$ to $x^2 + y^2 = 1$. (bonus: what if we change $\mathbb{R}$ to $\mathbb{F}_5$? $\mathbb{F}_p$?.)

$\sigma_1 = (12) = \sigma_2$ F (b) For any $\sigma_1, \sigma_2 \in S_n$, we have $l(\sigma_1 \circ \sigma_2) = l(\sigma_1) + l(\sigma_2)$.

3 vectors in $\mathbb{R}^2$ F (c) The 3 vectors $(1,1,0), (2,3,0), (4,5,0)$ in $\mathbb{R}^3$ are linearly independent.

cofactors are integers -T (d) If A is an $n \times n$ matrix with determinant 1 and integer entries, then A^{-1} also has integer entries.

T (e) If $z \in \mathbb{C}$ satisfies that $\bar{z}^2 = 1/z^3$, then $|z| = 1$. take modulus on both sides.

2. (50 points, 10 points each)

(a)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = ? \qquad \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

(b)

$$\det \begin{pmatrix} 0 & e & d \\ a & b & 0 \\ g & 0 & f \end{pmatrix} = ? \qquad -aef - gbd.$$

(c) Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear map of taking the derivative,

$$T(f(x)) = f'(x).$$

Is T surjective? Is T injective? (bonus: what if we change $\mathbb{R}$ to $\mathbb{F}_5$? or $\mathbb{F}_p$? can you write down $\ker(T), im(T)$?)

surjective ✓
not injective.
$T(1) = 0$
const function.

(d) Let $V = Map(\mathbb{R}, \mathbb{R})$ be the linear space of all functions on $\mathbb{R}$ and let $T : V \to \mathbb{R}$ be a linear map. Suppose we know that,

$$T(\sin(x)) = 1, \quad T(\cos(x)) = 3,$$

then for any given $\theta \in \mathbb{R}$,

$$T(\sin(x + \theta)) = ?$$

$T(\sin x \cdot \cos\theta + \cos x \cdot \sin\theta)$
$= \cos\theta + 3\sin\theta$

You may use $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.

(e) Let p be any prime (if you wish, you can let $p = 11$), let $S = \{(x, y, z) \in \mathbb{F}_p^3 \mid 2x + 3y + 5z = 1\}$. Is S a linear subspace of $\mathbb{F}_p^3$? What is the size of S?

NO

p^2

· say $p \neq 5$, one can choose x, y freely in $\mathbb{F}_p$ then z is determined, since 5 is invertible in $\mathbb{F}_p$.

· say $p = 5$, do the same thing for y, z. then x is determined.

1(a): False. $x^2 - y^2 = 1$ is a hyperbola.) (

$x^2 + y^2 = 1$ is a circle ○.

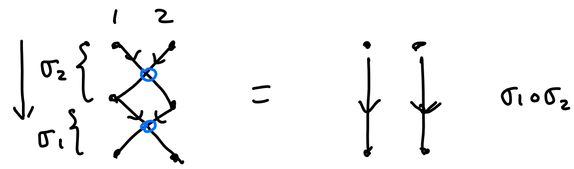
A linear transformation can only shear, flip, stretch the plane. So, it can not move one to another.

1(b) False. For example. let $n = 12$, $\sigma_1 = \binom{12}{21}$, $\sigma_2 = \binom{12}{21}$

$\sigma_1 \circ \sigma_2 = \binom{12}{12}$, $l(\sigma_1 \circ \sigma_2) = 0$, $l(\sigma_1) = l(\sigma_2) = 1$

$0 \neq 1 + 1$.

Graphically



$l = 2$ $\quad\quad$ $L = 0$.

1(c). False. span of the 3 vectors is contained in the z-plane $\{(x, y, z) \mid z = 0\}$, which is 2-dimensional. 3 vectors in 2 dimension space have to be linearly dependent.

(no need to find the actual linear combination to show linear dependence.)

1(d). True.

$$(A^{-1})_{ij} = \frac{1}{\det A} \cdot C_{ji} \overset{\because \det A = 1}{=} C_{ji}$$

C_{ji} is the co-factor. If A is an integer matrix, then any cofactor $C_{ij} \in \mathbb{Z}$.

1 (e) True. This one is best solved by applying the absolute sign $|\cdot|$ on both sides.

$$|\bar{z}^2| = |1/z^3|$$

but $|\bar{z}^2| = |\bar{z}|^2 = |z|^2$, $|1/z^3| = 1/|z^3| = 1/|z|^3$

Hence

$$|z|^2 = 1/|z|^3 \Rightarrow |z|^5 = 1$$

Since $|z| \geq 0$ $\therefore$ $|z| = 1$ is the only possibility.

2. (a), $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(b) $-aef - gbd$

most people get these 2 correct.

(c). T is surjective. To show this, let's fix a basis of $\mathbb{R}[x]$, as $\{1, x, x^2, x^3, \cdots\}$. If any basis vectors are in $\mathrm{im}(T)$, then the entire $\mathbb{R}[x] \subset \mathrm{im}(T)$, hence $\mathbb{R}[x] = \mathrm{im}(T)$. For x^n, we may see $\frac{1}{n} x^{n+1} \longmapsto x^n$ under T.

T is not injective. $\ker(T) = \mathbb{R} \cdot 1$, all the constant functions.

Here, a common mistake is about mistaking the injectivity for $T: \mathbb{R}[x] \to \mathbb{R}[x]$. for injectivity of a function $f: \mathbb{R} \to \mathbb{R}$.

$\mathbb{R}[x]$ is the set of all polynomial functions on $\mathbb{R}$. A particular function, say, $f(x) = 2x+2$, is only an element in $\mathbb{R}[x]$. We don't care $f(x)$, as a function $\mathbb{R} \to \mathbb{R}$, is surjective or injective here.

(d). $T(\sin(x+\theta))$

$$= T(\sin x \cdot \cos\theta + \cos x \cdot \sin\theta)$$

$$= 1 \cdot \cos\theta + 3 \cdot \sin\theta$$

as coefficient.

some mistakes are : keep a left-over T

like $1 \cdot T\cos\theta + 3 \cdot T\sin\theta$.

(e) • S is not a linear subspace, For example, $(0,0,0) \notin S$.

• Size of S is p^2.