Math H54 Midterm 1 Nov 14, 2022

Name: ______________________________

I_n denote the identity matrix of size n. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with complex entries. The standard Hermitian form on $\mathbb{C}^n$ is given by

$$\langle z, w \rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_n w_n$$

where $z = (z_1, \cdots, z_n)^t, w = (w_1, \cdots, w_n)^t$ are two column vectors. (superscript t stands for 'transpose')

1. (30 pts, 10 points each) True or False. If you think the answer is true, please justify your answer; if you think the answer is false, give a counter-example.

   (a) For any complex $n \times n$ matrix A, there exists an invertible $n \times n$ matrix C, such that $C^{-1}AC$ is a diagonal matrix.

   (b) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation, and $\mathbb{C}^n$ be equipped with the standard Hermitian form. Assume there exists a basis of $\mathbb{C}^n$ consist of eigenvectors of T, then these basis vectors are orthogonal to each other.

   (c) If T and S are linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, and $TS = ST$. Then if $Tv = \lambda v$ for some $v \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, we also have $T(Sv) = \lambda Sv$.

2. (35 points) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear map, where

$$\begin{aligned} Te_1 &= e_1 + e_2 \\ Te_2 &= e_2 + e_3 \\ Te_3 &= e_3 + e_1 \end{aligned}$$

(a) (10 points): If $z = (2, 3, 1)^t$, what is $Tz = ?$.

(b) (10 points): If $z = (z_1, z_2, z_3)^t$, what is $Tz = ?$

(c) (15 points): Can T be diagonalized? Namely, can you find a basis $\widetilde{e}_i$ of $\mathbb{C}^3$, such that $T\widetilde{e}_i = \lambda_i \widetilde{e}_i$ for some $\lambda_i \in \mathbb{C}$? Please give your reasoning. (Hint: you don't have to find $\widetilde{e}_i$ and λ_i explicitly.)

3. (35 points) Let $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a symmetric bilinear form, for any $z, w \in \mathbb{C}^n$, we have

$$B(z, w) = z^t[B]w = \sum_{i=1}^{n}\sum_{j=1}^{n} z_i B_{ij} w_j,$$

where $[B]$ is a symmetric matrix with entries B_{ij}.

(a) (10 points) if $\det([B]) \neq 0$, can you always find an invertible matrix C, such that $C^t[B]C = I_n$? Please explain.

(b) (10 points) if $\det([B]) \neq 0$, is it true that for any $v \neq 0$, we have $B(v, v) \neq 0$? Please explain.

(c) (15 points) Let the matrix $[B]$ be given by

$$[B] = \begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}$$

Find a matrix C such that $C^t[B]C$ is diagonal. Please show your steps.